

Diagonal Ramsey numbers of loose cycles in uniform hypergraphs

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Abstract

A k -uniform loose cycle \mathcal{C}_n^k is a hypergraph with vertex set $\{v_1, v_2, \dots, v_{n(k-1)}\}$ and with the set of n edges $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \dots, v_{(i-1)(k-1)+k}\}$, $1 \leq i \leq n$, where we use mod $n(k-1)$ arithmetic. The Ramsey number $R(\mathcal{C}_n^k, \mathcal{C}_n^k)$ is asymptotically $\frac{1}{2}(2k-1)n$ as has been proved by Gyárfás, Sárközy and Szemerédi. In this paper, we investigate to determining the exact value of diagonal Ramsey number of \mathcal{C}_n^k and we show that for $n \geq 2$ and $k \geq 8$

$$R(\mathcal{C}_n^k, \mathcal{C}_n^k) = (k-1)n + \lfloor \frac{n-1}{2} \rfloor.$$

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1 Introduction

For given k -uniform hypergraphs \mathcal{H}_1 and \mathcal{H}_2 , the *Ramsey number* $R(\mathcal{H}_1, \mathcal{H}_2)$ is the smallest number N such that in every red-blue coloring of the edges of the complete k -uniform hypergraph \mathcal{K}_N^k there is a red copy of \mathcal{H}_1 or a blue copy of \mathcal{H}_2 . A k -uniform loose path \mathcal{P}_n^k (shortly, a *path of length n*) is a hypergraph with vertex set $\{v_1, v_2, \dots, v_{n(k-1)+1}\}$ and with the set of n edges $e_i = \{v_1, v_2, \dots, v_k\} + (i-1)(k-1)$, $i = 1, 2, \dots, n$. For an edge e of a given loose path (also a given loose cycle) \mathcal{K} , the first vertex and the last vertex are denoted by $f_{\mathcal{K},e}$ and $l_{\mathcal{K},e}$, respectively. For $k = 2$ we get the usual definitions of a cycle C_n and a path P_n with n edges.

One of the most important problems in combinatorics and graph theory is determining or estimating the Ramsey numbers. In contrast to the graph case, there are few known results about the Ramsey numbers of hypergraphs. Recently, several interesting results were obtained on the Ramsey numbers of loose cycles in uniform hypergraphs. Haxell et al. [12] showed that the Ramsey number of 3-uniform loose cycles is asymptotically $\frac{5n}{2}$. More precisely, they proved that for all $\eta > 0$ there exists $n_0 = n_0(\eta)$ such that for every $n > n_0$, every 2-coloring of $\mathcal{K}_{\frac{5n}{2}(1+\eta)}^3$ contains

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a monochromatic copy of \mathcal{C}_n^3 . As extension of this result proved by Gyárfás, Sárközy and Szemerédi [11], shows that the Ramsey number of k -uniform loose cycles is asymptotically $\frac{1}{2}(2k-1)n$. However, those proofs are all based on the method of the Regularity Lemma. Concerning to exact values of Ramsey numbers of loose paths and cycles with at most 4 edges we have the following result.

Theorem 1.1 *For every $k \geq 3$,*

- (i) $R(\mathcal{C}_2^k, \mathcal{C}_2^k) + 1 = 2k - 2$ (to see a proof see Appendix A).
- (ii) ([9]) $R(\mathcal{P}_3^k, \mathcal{P}_3^k) = R(\mathcal{C}_3^k, \mathcal{P}_3^k) = R(\mathcal{C}_3^k, \mathcal{C}_3^k) + 1 = 3k - 1$.
- (iii) ([9]) $R(\mathcal{P}_4^k, \mathcal{P}_4^k) = R(\mathcal{C}_4^k, \mathcal{P}_4^k) = R(\mathcal{C}_4^k, \mathcal{C}_4^k) + 1 = 4k - 2$.

Regarding to Ramsey numbers of 3-uniform loose paths and cycles with arbitrary number of edges, in [9], the authors posed the following question.

Question 1.2 *For every $n \geq m \geq 3$, is it true that*

$$R(\mathcal{P}_n^3, \mathcal{P}_m^3) = R(\mathcal{P}_n^3, \mathcal{C}_m^3) = R(\mathcal{C}_n^3, \mathcal{C}_m^3) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor?$$

In particular, is it true that

$$R(\mathcal{P}_n^3, \mathcal{P}_n^3) = R(\mathcal{C}_n^3, \mathcal{C}_n^3) + 1 = \left\lceil \frac{5n}{2} \right\rceil?$$

Recently, this question is answered positively (see [14] and [15]). In [15] the authors posed the following conjecture on the values of Ramsey numbers of k -uniform loose paths and cycles for $k \geq 3$.

Conjecture 1.3 *Let $k \geq 3$ be an integer number. For every $n \geq m \geq 3$,*

$$R(\mathcal{P}_n^k, \mathcal{P}_m^k) = R(\mathcal{P}_n^k, \mathcal{C}_m^k) = R(\mathcal{C}_n^k, \mathcal{C}_m^k) + 1 = (k-1)n + \left\lfloor \frac{m+1}{2} \right\rfloor. \quad (1)$$

In [15], the authors also demonstrated that

Theorem 1.4 *For $n \geq m$, if $R(\mathcal{C}_n^k, \mathcal{C}_m^k) = (k-1)n + \left\lfloor \frac{m-1}{2} \right\rfloor$, then*

$$R(\mathcal{P}_n^k, \mathcal{P}_m^k) = R(\mathcal{P}_n^k, \mathcal{C}_m^k) = (k-1)n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

So they digested Conjecture 1.3 to the following conjecture.

Conjecture 1.5 *Let $k \geq 3$ be an integer number. For every $n \geq m \geq 3$,*

$$R(\mathcal{C}_n^k, \mathcal{C}_m^k) = (k-1)n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

In this paper, we investigate Conjecture 1.5 for the case $n = m$. More precisely, we prove the following theorem.

Theorem 1.6 *For every integers $k \geq 8$ and $n \geq 2$*

$$R(\mathcal{C}_n^k, \mathcal{C}_n^k) = (k-1)n + \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Using Theorems 1.4 and 1.6 we conclude the following result.

Theorem 1.7 *For every integers $k \geq 8$ and $n \geq 3$*

$$R(\mathcal{P}_n^k, \mathcal{P}_n^k) = R(\mathcal{P}_n^k, \mathcal{C}_n^k) = R(\mathcal{C}_n^k, \mathcal{C}_n^k) + 1 = (k-1)n + \lfloor \frac{n+1}{2} \rfloor.$$

Using Lemma 1 of [9], we have $R(\mathcal{C}_n^k, \mathcal{C}_m^k) \geq (k-1)n + \lfloor \frac{m-1}{2} \rfloor$. Therefore, in order to prove Theorem 1.6, it suffices to verify that the known lower bound are also upper bound.

Throughout the paper, we denote by \mathcal{H}_{red} and $\mathcal{H}_{\text{blue}}$ the induced k -uniform hypergraphs on the edges of \mathcal{H} with color red and blue, respectively. Also we denote by $|\mathcal{H}|$ and $\|\mathcal{H}\|$ the number of vertices and edges of \mathcal{H} , respectively.

The rest of this paper is organized as follows. In the next section, we give an outline of the proof of Theorem 1.6. In Section 3, we state some definitions and key technical lemmas required to prove the main theorem. For the sake of clarity of presentation, we omit the proofs in Section 3 and we refer the reader to Appendix A to see the complete proofs. We will present a complete proof of Theorem 1.6 in Section 4. In section 5, we conclude with some further remarks and open problems.

2 Outline of the proof of Theorem 1.6

In this section, we sketch the main idea of our proof for Theorem 1.6. We give a proof by induction on the number of vertices. By Theorem 1.1 we may assume that $n \geq 5$. Let $f(n) = (k-1)n + \lfloor \frac{n-1}{2} \rfloor$ and suppose to the contrary that $\mathcal{H} = \mathcal{K}_{f(n)}^k$ is 2-edge colored red and blue with no monochromatic copy of \mathcal{C}_n^k . We consider three following cases.

Case 1: $n \equiv 1, 2, 3 \pmod{4}$

First we show that there are two monochromatic copies of $\mathcal{C}_{\lfloor \frac{n}{2} \rfloor + 1}^k$ of colors red and blue (see Claims 4.1 and 4.8). Then among all red-blue copies of $\mathcal{C}_{\lfloor \frac{n}{2} \rfloor + 1}^k$'s choose red-blue copies with maximum intersection, say \mathcal{C}_1 and \mathcal{C}_2 . It is not difficult to see that

$$|V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq R(\mathcal{C}_{\lfloor \frac{n}{2} \rfloor + 1}^k, \mathcal{C}_{\lfloor \frac{n}{2} \rfloor + 1}^k) + 1.$$

Clearly $|V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)| \geq f(n-1 - \lfloor \frac{n}{2} \rfloor)$. So, by induction hypothesis, there is a monochromatic $\mathcal{C}_{n-1-\lfloor \frac{n}{2} \rfloor}^k$, say $\mathcal{C}_3 = e_1 e_2 \dots e_{n-1-\lfloor \frac{n}{2} \rfloor}$, disjoint from \mathcal{C}_1 and \mathcal{C}_2 . By symmetry we may assume that \mathcal{C}_2 and \mathcal{C}_3 are both in $\mathcal{H}_{\text{blue}}$ and $\mathcal{C}_1 \subseteq \mathcal{H}_{\text{red}}$.

Since there is no red copy of \mathcal{C}_n^k , we can find a red copy of $\mathcal{C}_{n-1-\lfloor \frac{n}{2} \rfloor}^k$, say $\mathcal{C}_4 = h_1 h_2 \dots h_{n-1-\lfloor \frac{n}{2} \rfloor}$, so that for each $1 \leq i \leq n-1-\lfloor \frac{n}{2} \rfloor$, $k-2 \leq |h_i \cap e_i| \leq k-1$. For even n , this follows from Claims 4.17 and 4.18. For odd n we use Claims 4.4 and 4.5. In the rest of the proof, we find a red copy of \mathcal{C}_n^k by joining \mathcal{C}_1 and \mathcal{C}_4 . If $n = 6$, we do this by applying Claims 4.10, 4.18 and 4.19. Otherwise, we apply Claims 4.4, 4.5 and 4.6 when $n \equiv^4 1, 3$ and Claims 4.10, 4.17 and 4.19 when $n \equiv^4 2$. This is a contradiction to our assumption. So we are done.

Case 2: $n \equiv 0 \pmod{4}$

In this case, first we show that there are two disjoint isochromatic $\mathcal{C}_{\frac{n}{2}}^k$. By symmetry we can suppose that $\mathcal{C}_1 = g_1 g_2 \dots g_{\frac{n}{2}}$ and $\mathcal{C}_2 = h_1 h_2 \dots h_{\frac{n}{2}}$ are such cycles in $\mathcal{H}_{\text{blue}}$. As we assumed that there is no blue copy of \mathcal{C}_n^k , we make a copy of \mathcal{C}_n^k in \mathcal{H}_{red} as follows. Let $W = V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)$. Since $n \geq 8$, we have $|W| \geq 3$. Use Lemma 3.8 for $e_i = g_1$, $f_j = h_1$ and $B = \{w_1, w_2, w_3\} \subseteq W$ to obtain two red paths E_1 and F_1 with desired properties. Assume that

$$g_i = \{x_1, x_2, \dots, x_k\} + (k-1)(i-1) \pmod{(k-1)\frac{n}{2}}, \quad i = 1, 2, \dots, \frac{n}{2},$$

$$h_i = \{y_1, y_2, \dots, y_k\} + (k-1)(i-1) \pmod{(k-1)\frac{n}{2}}, \quad i = 1, 2, \dots, \frac{n}{2}.$$

Now, use Lemma 3.9 for $e_i = g_2$, $f_j = h_2$ and $\mathcal{E}_1 = E_1$ (resp. $\mathcal{E}_1 = F_1$) to obtain two red paths \mathcal{E}_2 and \mathcal{F}_2 (resp. $\overline{\mathcal{E}}_2$ and $\overline{\mathcal{F}}_2$) of length 4 with mentioned properties of Lemma 3.9 (this can be done by a suitable renaming of the edges of \mathcal{C}_1 and \mathcal{C}_2). In the next step, for $2 \leq l \leq \frac{n}{2} - 2$, use Lemma 3.10 for red paths \mathcal{E}_l , \mathcal{F}_l (resp. $\overline{\mathcal{E}}_l$ and $\overline{\mathcal{F}}_l$), $e_i = g_{l+1}$ and $f_j = h_{l+1}$ to obtain two red paths \mathcal{E}_{l+1} and \mathcal{F}_{l+1} (resp. $\overline{\mathcal{E}}_{l+1}$ and $\overline{\mathcal{F}}_{l+1}$) with properties of Lemma 3.10. Note that each of $\mathcal{E}_{\frac{n}{2}-1}$, $\mathcal{F}_{\frac{n}{2}-1}$, $\overline{\mathcal{E}}_{\frac{n}{2}-1}$ and $\overline{\mathcal{F}}_{\frac{n}{2}-1}$ has $n-2$ edges.

Let $\mathcal{E}_{\frac{n}{2}-1} = p_1 p_2 \dots p_{n-2}$, $x \in g_{\frac{n}{2}-1} \setminus (\mathcal{E}_{\frac{n}{2}-1} \cup \{f_{\mathcal{C}_1, g_{\frac{n}{2}-1}}\})$, $y' \in (h_1 \setminus \{y_k\}) \cap (p_1 \setminus p_2)$ and $y'' \in (h_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_2, h_{\frac{n}{2}-1}}\}) \cap (p_{n-2} \setminus p_{n-3})$. Now, we define two edges q and q' as follows. Let

$$q = \{x, x_{(k-1)(\frac{n}{2}-1)+2}, \dots, x_{(k-1)(\frac{n}{2}-1)+\lfloor \frac{k}{2} \rfloor}\} \cup \{y_{(k-1)\frac{n}{2}-\lfloor \frac{k}{2} \rfloor+2}, \dots, y_{(k-1)\frac{n}{2}}, y'\},$$

and

$$q' = \{x_{(k-1)(\frac{n}{2}-1)+\lfloor \frac{k}{2} \rfloor}, x_{(k-1)(\frac{n}{2}-1)+\lfloor \frac{k}{2} \rfloor+2}, x_{(k-1)(\frac{n}{2}-1)+\lfloor \frac{k}{2} \rfloor+3}, \dots, x_{(k-1)\frac{n}{2}}, v\} \\ \cup \{y'', y_{(k-1)(\frac{n}{2}-1)+2}, \dots, y_{(k-1)(\frac{n}{2}-1)+\lfloor \frac{k}{2} \rfloor}\}.$$

If at least one of q or q' is blue, then we can find either a red copy of \mathcal{C}_n^k (using the red paths $\mathcal{E}_{\frac{n}{2}-1}$, $\mathcal{F}_{\frac{n}{2}-1}$, $\overline{\mathcal{E}}_{\frac{n}{2}-1}$ and $\overline{\mathcal{F}}_{\frac{n}{2}-1}$) or a blue copy of \mathcal{C}_n^k (using the blue cycles \mathcal{C}_1 and \mathcal{C}_2). Hence, $\mathcal{E}_{\frac{n}{2}-1} q' q$ is a red copy of \mathcal{C}_n^k . This contradicts our assumption.

3 Preliminaries

In this section, we present some results that will be used later on. For the clarity of presentation, except Lemmas 3.7 and 3.11 (their proofs are completely similar to the proofs of Lemmas 3.6 and 3.10, respectively), the proofs of all results in this section will be given in Appendix A.

Lemma 3.1 Let $n \geq 3$, $k \geq 6$, $f \geq (k-1)n$ and $\mathcal{H} = \mathcal{K}_f^k$ be 2-edge colored red and blue. If there is a $\mathcal{C}_n^k \subseteq \mathcal{H}_{\text{red}}$ and there is no red copy of \mathcal{C}_{n-1}^k , then $\mathcal{C}_n^k \subseteq \mathcal{H}_{\text{blue}}$.

Lemma 3.2 Let $n \geq 3$, $k \geq 6$, $f \geq (k-1)n$ and $\mathcal{H} = \mathcal{K}_f^k$ be 2-edge colored red and blue. If there is a $\mathcal{C}_n^k \subseteq \mathcal{H}_{\text{red}}$ and there is no red copy of \mathcal{C}_{n-2}^k , then $\mathcal{C}_n^k \subseteq \mathcal{H}_{\text{blue}}$.

Remark 3.3 In the sequel of this section assume that $n \geq 5$, $k \geq 8$, $l_2 \geq l_1 \geq 2$, $l_1 + l_2 \leq n$ and $\mathcal{H} = \mathcal{K}_{(k-1)n + \lfloor \frac{n-1}{2} \rfloor}^k$ is 2-edge colored red and blue. Also, let $\mathcal{C}_1 = e_1 e_2 \dots e_{l_1}$ and $\mathcal{C}_2 = f_1 f_2 \dots f_{l_2}$ be two disjoint cycles in $\mathcal{H}_{\text{blue}}$ with edges

$$e_i = \{v_1, v_2, \dots, v_k\} + (k-1)(i-1) \pmod{(k-1)l_1}, \quad i = 1, \dots, l_1$$

and

$$f_i = \{u_1, u_2, \dots, u_k\} + (k-1)(i-1) \pmod{(k-1)l_2}, \quad i = 1, \dots, l_2,$$

respectively, and W with $|W| \geq 2$ be the set of vertices uncovered by $\mathcal{C}_1 \cup \mathcal{C}_2$.

Let e_i and f_j be two edges of \mathcal{C}_1 and \mathcal{C}_2 , respectively and $g = E \dot{\cup} W' \dot{\cup} F$, where

- $E = E' \cup \{v'\}$ for $v' \in (e_{i-1} \setminus \{f_{\mathcal{C}_1, e_{i-1}}\}) \cup (e_{i+1} \setminus \{l_{\mathcal{C}_1, e_{i+1}}\})$ and $E' \subseteq V(e_i) \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\}$.
- $F = F' \cup \{u'\}$ for $u' \in (f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}) \cup (f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\})$ and $F' \subseteq V(f_j) \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\}$.
- $W' \subseteq W$.
- $|E| = p \geq 1, |W'| = q \geq 0, |F| = r \geq 1$ and $p + q + r = k$.

We say that g is of type A , B , C or D if $(v', u') \in (e_{i-1} \setminus \{f_{\mathcal{C}_1, e_{i-1}}\}) \times (f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\})$, $(v', u') \in (e_{i+1} \setminus \{l_{\mathcal{C}_1, e_{i+1}}\}) \times (f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\})$, $(v', u') \in (e_{i-1} \setminus \{f_{\mathcal{C}_1, e_{i-1}}\}) \times (f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\})$ or $(v', u') \in (e_{i+1} \setminus \{l_{\mathcal{C}_1, e_{i+1}}\}) \times (f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\})$, respectively. By \mathcal{A}_{ij} , \mathcal{B}_{ij} , \mathcal{C}_{ij} , \mathcal{D}_{ij} we mean all edges of type A , B , C , D corresponding to the edges e_i and f_j , respectively.

Remark 3.4 Consider the edges e_i and f_j . For $g \in \mathcal{A}_{ij}$ (resp. $g \in \mathcal{B}_{ij}$) and for every $v'' \in e_{i+1} \setminus (V(g) \cup \{l_{\mathcal{C}_1, e_{i+1}}\})$ (resp. $v'' \in e_{i-1} \setminus (V(g) \cup \{f_{\mathcal{C}_1, e_{i-1}}\})$) and $u'' \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}$ (resp. $u'' \in f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\}$), there is an edge $g' \in \mathcal{B}_{ij}$ (resp. $g' \in \mathcal{A}_{ij}$) where $\{u'', v''\} \subseteq g'$ and $g \cap g' = \emptyset$. The same result holds for the edges of types C and D .

Remark 3.5 If there is no blue copy of $\mathcal{C}_{l_1+l_2}$ in \mathcal{H} , then there are no two disjoint edges $g \in \mathcal{A}_{ij}$ (resp. $g \in \mathcal{C}_{ij}$) and $g' \in \mathcal{B}_{ij}$ (resp. $g' \in \mathcal{D}_{ij}$) of colors blue in \mathcal{H} .

Lemma 3.6 *With the same assumptions in Remark 3.3, let e_i and f_j be two arbitrary edges of \mathcal{C}_1 and \mathcal{C}_2 , respectively, and $C \subseteq \{v\}$, where $v \in e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\}$. Also, let $B = \{w_1, w_2\} \subseteq W$ and let $e \in \{e_i, f_j\}$ be an edge of \mathcal{C}_r for some $r \in \{1, 2\}$. Assume that v', v'', u' and u'' are distinct vertices so that $v' \in e_{i-1} \setminus \{f_{\mathcal{C}_1, e_{i-1}}\}$, $v'' \in e_{i+1} \setminus \{l_{\mathcal{C}_1, e_{i+1}}\}$, $u' \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}$ and $u'' \in f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\}$. If there is no blue copy of $\mathcal{C}_{l_1+l_2}$, then we can find a red path $\mathcal{P} = g_1 g_2$ so that for some vertex $w \in e \setminus (\{f_{\mathcal{C}_r, e}\} \cup C)$, we have $w \notin \mathcal{P}$ and the following conditions hold.*

- i) $V(\mathcal{P}) \subseteq \left(V(e_i) \setminus (C \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\}) \right) \cup \left(V(f_j) \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\} \right) \cup B' \cup \{v', v'', u', u''\}$
where $B' \subseteq B$ with $|B'| \leq |C| + 1$.
- ii) $|g_m \cap (e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})| \geq 2, |g_m \cap (f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\})| \geq 2$, for $m = 1, 2$.
- iii) If $|C| = 1$, then either $w_1 \in g_1 \setminus g_2$ and $w_2 \in g_2 \setminus g_1$ or $|B \cap g_1| = |B \cap g_2| + 1 = 1$.

By an argument similar to the proof of Lemma 3.6 in Appendix A, we have the following.

Lemma 3.7 *With the same assumptions in Remark 3.3, let e_i and f_j be two arbitrary edges of \mathcal{C}_1 and \mathcal{C}_2 , respectively. Assume that v', v'', u' and u'' are distinct vertices so that $v' \in e_{i-1} \setminus \{f_{\mathcal{C}_1, e_{i-1}}\}$, $v'' \in e_{i+1} \setminus \{l_{\mathcal{C}_1, e_{i+1}}\}$, $u' \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}$ and $u'' \in f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\}$. If there is no blue copy of $\mathcal{C}_{l_1+l_2}$, then we can find a red path $\mathcal{P} = g_1 g_2$ so that the following conditions hold.*

- $V(\mathcal{P}) \subseteq (V(e_i) \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\}) \cup (V(f_j) \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\}) \cup \{v', v'', u', u''\}$.
- $|g_m \cap (e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})| \geq 3, |g_m \cap (f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\})| \geq 3$, for $m = 1, 2$.

Lemma 3.8 *With the same assumptions in Remark 3.3, let e_i and f_j be two arbitrary edges of \mathcal{C}_1 and \mathcal{C}_2 , respectively. Also, let $B = \{w_1, w_2, w_3\} \subseteq W$. If there is no blue copy of $\mathcal{C}_{l_1+l_2}$, then we can find two red paths $E_1 = g_1 g'_1$ and $F_1 = g_1 \overline{g}_1$ so that the following conditions hold.*

- $V(E_1), V(F_1) \subseteq \left(V(e_i) \cup V(f_j) \cup B' \right) \setminus \{\overline{u}, \overline{v}\}$ for some $\overline{v} \in e_i$, $\overline{u} \in f_j$ and $B' \subseteq B$ with $|B'| = 2$.
- $e_i \setminus (V(E_1) \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}, \overline{v}\}) \neq \emptyset$ and $f_j \setminus (V(F_1) \cup \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}, \overline{u}\}) \neq \emptyset$.
- $|g'_1 \setminus g_1 \cap (e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})| \geq 1, |(g'_1 \setminus g_1) \cap (f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\})| \geq 1$.
- $|g_1 \setminus g'_1 \cap (e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})| \geq 1, |(g_1 \setminus g'_1) \cap (f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\})| \geq 1$.
- $|(\overline{g}'_1 \setminus g_1) \cap (e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})| \geq 1, |(\overline{g}'_1 \setminus g_1) \cap (f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\})| \geq 1$.
- $|g_1 \setminus \overline{g}'_1 \cap (e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})| \geq 1, |(g_1 \setminus \overline{g}'_1) \cap (f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\})| \geq 1$.
- $|B' \cap (g_1 \setminus g'_1)| = |B' \cap (g'_1 \setminus g_1)| = |B' \cap (g_1 \setminus \overline{g}'_1)| = |B' \cap (\overline{g}'_1 \setminus g_1)| = 1$.

Lemma 3.9 *With the same assumptions in Remark 3.3, let $l_1 \geq 3$ and e_i and f_j be two arbitrary edges of \mathcal{C}_1 and \mathcal{C}_2 , respectively. Also, let $\mathcal{E}_1 = E_1 = g_1 g'_1$ be a red path of length 2 with the following properties.*

- $V(E_1) \subseteq (V(e_{i-1}) \cup V(f_{j-1}) \cup B') \setminus \{\bar{u}, \bar{v}\}$ where $\bar{v} \in e_{i-1} \setminus \{f_{C_1, e_{i-1}}\}$, $\bar{u} \in f_{j-1} \setminus \{f_{C_2, f_{j-1}}\}$ and $B' \subseteq W$ with $|B'| = 2$.
- $|(g'_1 \setminus g_1) \cap (e_{i-1} \setminus \{f_{C_1, e_{i-1}}, l_{C_1, e_{i-1}}\})| \geq 1$, $|(g'_1 \setminus g_1) \cap (f_{j-1} \setminus \{f_{C_2, f_{j-1}}, l_{C_2, f_{j-1}}\})| \geq 1$.
- $(g'_1 \setminus g_1) \cap B' = \{w\}$.

If there is no blue copy of $C_{l_1+l_2}$, then there are two red paths $E_2 = g_2 g'_2$ and $F_2 = \bar{g}_2 \bar{g}'_2$ of length 2 so that the following conditions hold.

- $V(E_2), V(F_2) \subseteq ((V(e_i) \cup V(f_j)) \setminus \{f_{C_1, e_i}, f_{C_2, f_j}\}) \cup \{w, \hat{u}, \hat{v}\}$ for some $\hat{v} \in (e_{i-1} \setminus \{f_{C_1, e_{i-1}}\}) \cap (V(E_1) \cup \{\bar{v}\})$, $\hat{u} \in (f_{j-1} \setminus \{f_{C_2, f_{j-1}}\}) \cap (V(E_1) \cup \{\bar{u}\})$.
- $e_i \setminus (E_2 \cup \{f_{C_1, e_i}\}) \neq \emptyset$ and $f_j \setminus (F_2 \cup \{f_{C_2, f_j}\}) \neq \emptyset$.
- $|(g'_2 \setminus g_2) \cap (e_i \setminus \{f_{C_1, e_i}, l_{C_1, e_i}\})| \geq 1$, $|(\bar{g}'_2 \setminus \bar{g}_2) \cap (f_j \setminus \{f_{C_2, f_j}, l_{C_2, f_j}\})| \geq 1$.
- $\mathcal{E}_2 = \mathcal{E}_1 E_2$ and $\mathcal{F}_2 = \mathcal{F}_1 F_2$ are two red paths of length 4.

Lemma 3.10 *With the same assumptions in Remark 3.3, let $i \geq 2$ and*

$$\mathcal{E}_{i-1} = E_1 E_2 \dots E_{i-1}, \mathcal{F}_{i-1} = F_1 F_2 \dots F_{i-1}$$

be two red paths of length $2i-2$ where for each $1 \leq t \leq i-1$, $E_t = g_t g'_t$ and $F_t = \bar{g}_t \bar{g}'_t$ are red paths of length 2 with the following properties.

P_1 : *For each $1 \leq t \leq i-1$,*

$$V(E_t), V(F_t) \subseteq (V(e_t) \setminus \{f_{C_1, e_t}\}) \cup (V(f_t) \setminus \{f_{C_2, f_t}\}) \cup B_t \cup \{\hat{v}_t, \hat{u}_t\}$$

where $\hat{v}_t \in e_{t-1} \setminus \{f_{C_1, e_{t-1}}\}$, $\hat{u}_t \in f_{t-1} \setminus \{f_{C_2, f_{t-1}}\}$, $|B_t| \leq 2$ for $t = 1$ and $|B_t \setminus \bigcup_{j=1}^{t-1} B_j| \leq 1$ for $t \neq 1$.

P_2 : *For $t = i-1$*

- $e_t \setminus (E_t \cup \{f_{C_1, e_t}\}) \neq \emptyset$ and $f_t \setminus (F_t \cup \{f_{C_2, f_t}\}) \neq \emptyset$.
- $|(g'_t \setminus g_t) \cap (e_t \setminus \{f_{C_1, e_t}, l_{C_1, e_t}\})| \geq 1$, $|(g'_t \setminus g_t) \cap (f_t \setminus \{f_{C_2, f_t}, l_{C_2, f_t}\})| \geq 1$.
- $|(\bar{g}'_t \setminus \bar{g}_t) \cap (e_t \setminus \{f_{C_1, e_t}, l_{C_1, e_t}\})| \geq 1$, $|(\bar{g}'_t \setminus \bar{g}_t) \cap (f_t \setminus \{f_{C_2, f_t}, l_{C_2, f_t}\})| \geq 1$.

If there is no blue copy of $C_{l_1+l_2}$ and $W \setminus \bigcup_{t=1}^{i-1} B_t \neq \emptyset$, then there are two red paths $E_i = g_i g'_i$ and $F_i = \bar{g}_i \bar{g}'_i$ of length 2 such that the properties P_1 and P_2 hold for $t = i$ and for some $\mathcal{P} \in \{\mathcal{E}_{i-1}, \mathcal{F}_{i-1}\}$, $\mathcal{E}_i = \mathcal{P} E_i$ and $\mathcal{F}_i = \mathcal{P} F_i$ are two red paths of length $2i$.

The proof of the following Lemma is similar to the proof of Lemma 3.10.

Lemma 3.11 *Let $\mathcal{E}_{i-1} = E_1 E_2 \dots E_{i-1}$ and $\mathcal{F}_{i-1} = F_1 F_2 \dots F_{i-1}$ be two red paths of length $2i-2$ with the same properties of Lemma 3.10. Let $v \in e_{i+1} \setminus \{l_{C_1, e_{i+1}}\}$ and $u \in f_{i+1} \setminus \{l_{C_2, f_{i+1}}\}$ are two distinct vertices so that $\{v, u\} \cap V(\mathcal{E}_{i-1} \cup \mathcal{F}_{i-1}) = \emptyset$. If there is no blue copy of $C_{l_1+l_2}$, then there is a red path $E_i = g_i g'_i$ of length 2 such that the following conditions hold.*

- $V(E_i) \subseteq (V(e_i) \setminus \{f_{C_1, e_i}, l_{C_1, e_i}\}) \cup (V(f_i) \setminus \{f_{C_2, f_i}, l_{C_2, f_i}\}) \cup \{u, v, u', v'\}$ where $v' \in e_{i-1} \setminus \{f_{C_1, e_{i-1}}\}$ and $u' \in f_{i-1} \setminus \{f_{C_2, f_{i-1}}\}$.
- $|(g'_i \setminus g_i) \cap (e_i \setminus \{f_{C_1, e_i}, l_{C_1, e_i}\})| \geq 2$, $|(g'_i \setminus g_i) \cap (f_i \setminus \{f_{C_2, f_i}, l_{C_2, f_i}\})| \geq 2$.
- For some $\mathcal{P} \in \{\mathcal{E}_{i-1}, \mathcal{F}_{i-1}\}$, $\mathcal{E}_i = \mathcal{P}E_i$ is a red path of length $2i$.

Lemma 3.12 Let $n \geq 5$ be odd, $k \geq 8$ and $\mathcal{H} = \mathcal{K}_{(k-1)n + \lfloor \frac{n-1}{2} \rfloor}^k$ be 2-edge colored red and blue. Let \mathcal{C}_1 and \mathcal{C}_2 be two disjoint blue cycles of length $\frac{n-1}{2}$ and $\frac{n+1}{2}$, respectively. If there is no blue copy of \mathcal{C}_n^k , then there is a copy of $\mathcal{C}_{\frac{n+1}{2}}^k$ in \mathcal{H}_{red} .

Lemma 3.13 Let $n \geq 6$ be even, $k \geq 8$ and $\mathcal{H} = \mathcal{K}_{(k-1)n + \lfloor \frac{n-1}{2} \rfloor}^k$ be 2-edge colored red and blue. Let \mathcal{C}_1 and \mathcal{C}_2 be two disjoint blue cycles of length $\frac{n}{2} - 1$ and $\frac{n}{2} + 1$, respectively. If there is no blue copy of \mathcal{C}_n^k , then there is a copy of $\mathcal{C}_{\frac{n}{2}+1}^k$ in \mathcal{H}_{red} .

4 Proof of Theorem 1.6

We prove the theorem by induction on n . By Theorem 1.1 the theorem holds when $n \leq 4$. Let $f(n) = (k-1)n + \lfloor \frac{n-1}{2} \rfloor$ and suppose to the contrary that $\mathcal{H} = \mathcal{K}_{f(n)}^k$ is 2-edge colored red and blue with no monochromatic copy of \mathcal{C}_n^k . We consider the following cases.

Case 1: n is odd

Claim 4.1 There are two monochromatic copies of $\mathcal{C}_{\frac{n+1}{2}}^k$ of colors red and blue.

Proof. Since $f(\frac{n+1}{2}) < f(n)$, using induction hypothesis, there is a monochromatic copy of $\mathcal{C}_{\frac{n+1}{2}}^k$, say \mathcal{C}_1 . Because of the symmetry we may assume that $\mathcal{C}_1 \subseteq \mathcal{H}_{\text{blue}}$. Since $|V(\mathcal{C}_1)| = (k-1)(\frac{n+1}{2})$ and

$$f(\frac{n-1}{2}) = (k-1)(\frac{n-1}{2}) + \lfloor \frac{n-3}{4} \rfloor < f(n) - (k-1)(\frac{n+1}{2}),$$

there is a monochromatic $\mathcal{C}_2 = \mathcal{C}_{\frac{n-1}{2}}^k$ in $V(\mathcal{H}) \setminus V(\mathcal{C}_1)$. If $\mathcal{C}_2 \subseteq \mathcal{H}_{\text{blue}}$, using Lemma 3.12, there is a copy of $\mathcal{C}_{\frac{n+1}{2}}^k$ in \mathcal{H}_{red} and so we are done. Now let $\mathcal{C}_2 \subseteq \mathcal{H}_{\text{red}}$. If there is no $\mathcal{C}_{\frac{n-1}{2}}^k$ in $\mathcal{H}_{\text{blue}}$, then using Lemma 3.1 there is a red copy of $\mathcal{C}_{\frac{n+1}{2}}^k$ and again we are done. Therefore, we may assume that we have a red copy of $\mathcal{C}_{\frac{n-1}{2}}^k$ and a blue copy of $\mathcal{C}_{\frac{n-1}{2}}^k$. Among all red-blue copies of $\mathcal{C}_{\frac{n-1}{2}}^k$'s choose red-blue copies with maximum intersection, say \mathcal{C}'_1 and \mathcal{C}'_2 . Set $A = V(\mathcal{C}'_1) \cup V(\mathcal{C}'_2)$. We can show that $|A| \leq R(\mathcal{C}_{\frac{n-1}{2}}^k, \mathcal{C}_{\frac{n-1}{2}}^k) + 1$. To see that, suppose to the contrary that $|A| \geq R(\mathcal{C}_{\frac{n-1}{2}}^k, \mathcal{C}_{\frac{n-1}{2}}^k) + 2$. Since

$$|A| \geq \max\{|V(\mathcal{C}'_1)|, |V(\mathcal{C}'_2)|\} + 2,$$

then $B_1 = V(\mathcal{C}'_1) \setminus V(\mathcal{C}'_2)$ and $B_2 = V(\mathcal{C}'_2) \setminus V(\mathcal{C}'_1)$ are non-empty. Choose $v_1 \in B_1$ and $v_2 \in B_2$ and set

$$U = \left(V(\mathcal{C}'_1) \cup V(\mathcal{C}'_2) \right) \setminus \{v_1, v_2\}.$$

Since $|U| \geq R(\mathcal{C}_{\frac{n-1}{2}}, \mathcal{C}_{\frac{n-1}{2}})$, we have a monochromatic copy of $\mathcal{C}_{\frac{n-1}{2}}$, say \mathcal{C} . If \mathcal{C} is red, $|\mathcal{C} \cap \mathcal{C}'_2| > |\mathcal{C}'_1 \cap \mathcal{C}'_2|$. If \mathcal{C} is blue, then $|\mathcal{C} \cap \mathcal{C}'_1| > |\mathcal{C}'_1 \cap \mathcal{C}'_2|$. Both cases contradict the choices of \mathcal{C}'_1 and \mathcal{C}'_2 . So $|A| \leq R(\mathcal{C}_{\frac{n-1}{2}}^k, \mathcal{C}_{\frac{n-1}{2}}^k) + 1$. Clearly

$$f\left(\frac{n+1}{2}\right) \leq f(n) - |A|.$$

So using induction hypothesis, we have a monochromatic $\mathcal{C}_{\frac{n+1}{2}}^k$, say \mathcal{C} , in $V(\mathcal{H}) \setminus V(\mathcal{C}'_1 \cup \mathcal{C}'_2)$. If \mathcal{C} is red, then \mathcal{C} and \mathcal{C}_1 are desired monochromatic copies of $\mathcal{C}_{\frac{n+1}{2}}^k$. If not, since there is no blue \mathcal{C}_n , using Lemma 3.12, there is a copy of $\mathcal{C}_{\frac{n+1}{2}}^k$, say \mathcal{C}' , in \mathcal{H}_{red} . Clearly \mathcal{C}' and \mathcal{C}_1 are our favorable cycles. \square

Among all red-blue copies of $\mathcal{C}_{\frac{n+1}{2}}^k$'s choose red-blue copies with maximum intersection. Let $\mathcal{C}_1 \subseteq \mathcal{H}_{\text{red}}$ and $\mathcal{C}_2 \subseteq \mathcal{H}_{\text{blue}}$ be such copies. Assume that $\mathcal{C}_1 = d_1 d_2 \dots d_{\frac{n+1}{2}}$. An argument similar to the proof of Claim 4.1 yields

$$|V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq R(\mathcal{C}_{\frac{n+1}{2}}^k, \mathcal{C}_{\frac{n+1}{2}}^k) + 1.$$

Let $B = V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)$. It is easy to show that $|B| \geq f(\frac{n-1}{2})$. So, by induction hypothesis, there is a monochromatic $\mathcal{C}_{\frac{n-1}{2}}^k$, say \mathcal{C}_3 , in B . By symmetry we may assume that $\mathcal{C}_3 \subseteq \mathcal{H}_{\text{blue}}$. Let $\mathcal{C}_2 = f_1 f_2 \dots f_{\frac{n+1}{2}}$, $\mathcal{C}_3 = e_1 e_2 \dots e_{\frac{n-1}{2}}$ and $W = V(\mathcal{H}) \setminus V(\mathcal{C}_2 \cup \mathcal{C}_3)$ where

$$f_i = \{u_1, u_2, \dots, u_k\} + (k-1)(i-1) \pmod{(k-1)\left(\frac{n+1}{2}\right)}, \quad i = 1, 2, \dots, \frac{n+1}{2}$$

and

$$e_i = \{v_1, v_2, \dots, v_k\} + (k-1)(i-1) \pmod{(k-1)\left(\frac{n-1}{2}\right)}, \quad i = 1, 2, \dots, \frac{n-1}{2}.$$

Claim 4.2 *Let f_j be an arbitrary edge of \mathcal{C}_2 and $z \in W$. There are vertices $\bar{u} \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}$, $\bar{u}' \in f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\}$ and $\hat{u} \in f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\}$ so that the edge*

$$g = (f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}, \hat{u}\}) \cup \{\bar{u}, \bar{u}', z\}$$

is blue.

Proof of Claim 4.2. Suppose not. We may assume that $f_j = f_{\frac{n+1}{2}}$. We find a red copy of \mathcal{C}_n^k as follows.

Use Lemma 3.6 for $e = e_1$ (resp. $e = f_1$) (by putting $i = j = 1$, $v' = v_1$, $v'' = v_k$, $u' = u_{(k-1)(\frac{n+1}{2})}$, $u'' = u_k$, $C = \{v_{k-1}\}$ and $B = \{z, w_1\} \subseteq W$) to obtain a red

path $E_1 = g_1 g'_1$ (resp. $F_1 = \bar{g}_1 \bar{g}'_1$) with the mentioned properties of Lemma 3.6. Let $\mathcal{E}_1 = E_1$ and $\mathcal{F}_1 = F_1$. We may assume that $(g_1 \setminus g'_1) \cap \{z\} = E_1 \cap \{z\}$ (also $(\bar{g}_1 \setminus \bar{g}'_1) \cap \{z\} = F_1 \cap \{z\}$). Use Lemma 3.10, $\frac{n-5}{2}$ times to obtain two red paths $\mathcal{E}_{\frac{n-3}{2}}$ and $\mathcal{F}_{\frac{n-3}{2}}$ of length $n-3$ with desired properties.

Let $i = \frac{n-1}{2}$, $v = v_{k-1}$, $u = f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}$ and use Lemma 3.11 to obtain a red path $E_{\frac{n-1}{2}} = g_{\frac{n-1}{2}} g'_{\frac{n-1}{2}}$ so that for some $\mathcal{P} \in \{\mathcal{E}_{\frac{n-3}{2}}, \mathcal{F}_{\frac{n-3}{2}}\}$, $\mathcal{E}_{\frac{n-1}{2}} = \mathcal{P} E_{\frac{n-1}{2}}$ is a red path of length $n-1$. Let $\mathcal{E}_{\frac{n-1}{2}} = h_1 h_2 \dots h_{n-1}$, $\bar{u} \in (f_{\frac{n-1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-1}{2}}}\}) \cap (h_{n-1} \setminus h_{n-2})$ and $\hat{u} = u_{(k-1)(\frac{n+1}{2})}$. If $z \in h_1$, then set $\bar{u}' = u_1$. Otherwise, let $\bar{u}' \in (f_1 \setminus \{l_{\mathcal{C}_2, f_1}\}) \cap (h_1 \setminus h_2)$. Now set

$$g = (f_{\frac{n+1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}, l_{\mathcal{C}_2, f_{\frac{n+1}{2}}}, \hat{u}\}) \cup \{\bar{u}, \bar{u}', z\}.$$

Clearly, $\mathcal{E}_{\frac{n-1}{2}} g$ is a red copy of \mathcal{C}_n^k , a contradiction to our assumption. This contradiction completes the proof of Claim 4.2. \square

Claim 4.3 *Let e_i and f_j be two arbitrary edges of \mathcal{C}_3 and \mathcal{C}_2 , respectively. For every vertices $\bar{u}' \in f_j \setminus \{f_{\mathcal{C}_2, f_j}\}$ and $x \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}$ (resp. $\bar{u}' \in f_j \setminus \{l_{\mathcal{C}_2, f_j}\}$ and $x \in f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}\}$), there are vertices $\bar{v} \in e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}$ and $\bar{u} \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}, x\}$ (resp. $\bar{u} \in f_{j+1} \setminus \{l_{\mathcal{C}_2, f_{j+1}}, x\}$) so that the edge $(V(f_j) \setminus \{f_{\mathcal{C}_2, f_j}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\}$ (resp. the edge $(V(f_j) \setminus \{l_{\mathcal{C}_2, f_j}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\})$ is blue.*

Proof of Claim 4.3. By symmetry it only suffices to show that for every vertices $\bar{u}' \in f_j \setminus \{f_{\mathcal{C}_2, f_j}\}$ and $x \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}$, there are vertices $\bar{v} \in e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}$ and $\bar{u} \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}, x\}$ so that the edge

$$(V(f_j) \setminus \{f_{\mathcal{C}_2, f_j}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\}$$

is blue. With no loss of generality we may assume that $e_i = e_1$ and $f_j = f_{\frac{n+1}{2}}$. Suppose to the contrary that there are vertices $\bar{u}' \in f_{\frac{n+1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}\}$ and $x \in f_{\frac{n-1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-1}{2}}}\}$ such that for every vertices $\bar{v} \in e_1 \setminus \{v_1, v_k\}$ and $\bar{u} \in f_{\frac{n-1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-1}{2}}}, x\}$ the edge

$$(V(f_{\frac{n+1}{2}}) \setminus \{f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\}$$

is red. We can find a red copy of \mathcal{C}_n^k as follows.

Use Lemma 3.6 for $e = e_1$ (resp. $e = f_1$) to obtain a red path $E_1 = g_1 g'_1$ (resp. $F_1 = \bar{g}_1 \bar{g}'_1$) with the mentioned properties of Lemma 3.6 (by putting $i = j = 1$, $v' = v_1$, $v'' = v_k$, $u' = \bar{u}'$, $u'' = u_k$, $C = \{v_{k-1}\}$ and $B = \{w_1, w_2\} \subseteq W$). Let $\mathcal{E}_1 = E_1$ and $\mathcal{F}_1 = F_1$. Use Lemma 3.10, $\frac{n-5}{2}$ times to obtain two red paths $\mathcal{E}_{\frac{n-3}{2}}$ and $\mathcal{F}_{\frac{n-3}{2}}$ of length $n-3$.

Let $i = \frac{n-1}{2}$, $v = v_{k-1}$, $u = f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}$ and use Lemma 3.11 to obtain a red path $E_{\frac{n-1}{2}} = g_{\frac{n-1}{2}} g'_{\frac{n-1}{2}}$ so that for some $\mathcal{P} \in \{\mathcal{E}_{\frac{n-3}{2}}, \mathcal{F}_{\frac{n-3}{2}}\}$, $\mathcal{E}_{\frac{n-1}{2}} = \mathcal{P} E_{\frac{n-1}{2}}$ is a

red path of length $n - 1$. Let $\mathcal{E}_{\frac{n-1}{2}} = h_1 h_2 \dots h_{n-1}$, $\bar{v} \in (e_1 \setminus \{v_1, v_k\}) \cap (h_1 \setminus h_2)$, $\bar{u} \in (f_{\frac{n-1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-1}{2}}}, x\}) \cap (h_{n-1} \setminus h_{n-2})$ and

$$g = (V(f_{\frac{n+1}{2}}) \setminus \{f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\}.$$

Clearly, $\mathcal{E}_{\frac{n-1}{2}} g$ is a red copy of \mathcal{C}_n^k , a contradiction to our assumption. This contradiction completes the proof of Claim 4.3. \square

Claim 4.4 *Let e_i and f_j be two arbitrary edges of \mathcal{C}_3 and \mathcal{C}_2 , respectively. If $n \geq 7$, then for every vertices $z \in f_j$ and $v \in (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}) \cup (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\})$ the edge $(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, v\}$ is red. If $n = 5$, then for every $z \in f_j$ and $v \in \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}$ the edge*

$$(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, v\}$$

is red.

Proof of Claim 4.4. We give only a proof for $n \geq 7$. The proof for $n = 5$ is similar. Suppose for a contradiction that there are vertices $z \in f_j$ and $v \in (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}) \cup (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\})$ so that the edge

$$g = (V(e_i) \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, v\}$$

is blue. With no loss of generality assume that $v \in e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}$ (by symmetry the case $v \in e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\}$ is similar). If $z = l_{\mathcal{C}_2, f_j}$, then put $\bar{u}' = z$ and $x = l_{\mathcal{C}_2, f_{j-1}}$ and use Claim 4.3 to obtain a blue edge

$$g' = (V(f_j) \setminus \{f_{\mathcal{C}_2, f_j}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\}$$

for some $\bar{v} \in e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}, l_{\mathcal{C}_3, e_{i-1}}\}$ and $\bar{u} \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}, x\}$. Clearly

$$g' f_{j-1} f_{j-2} \dots f_1 f_{\frac{n+1}{2}} \dots f_{j+1} g e_{i+1} e_{i+2} \dots e_{\frac{n-1}{2}} e_1 \dots e_{i-1}$$

is a blue \mathcal{C}_n^k , a contradiction to our assumption. If $z \in f_j \setminus \{l_{\mathcal{C}_2, f_j}\}$, then put $\bar{u}' = l_{\mathcal{C}_2, f_{j-1}}$ and $x = l_{\mathcal{C}_2, f_{j-2}}$ and use Claim 4.3 to obtain a blue edge

$$g'' = (V(f_{j-1}) \setminus \{f_{\mathcal{C}_2, f_{j-1}}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\}$$

for some $\bar{v} \in e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}, l_{\mathcal{C}_3, e_{i-1}}\}$ and $\bar{u} \in f_{j-2} \setminus \{f_{\mathcal{C}_2, f_{j-2}}, x\}$. Again

$$g'' f_{j-2} f_{j-3} \dots f_1 f_{\frac{n+1}{2}} \dots f_j g e_{i+1} e_{i+2} \dots e_{\frac{n-1}{2}} e_1 \dots e_{i-1}$$

is a blue copy of \mathcal{C}_n^k , a contradiction to our assumption. \square

Claim 4.5 *Let e_i be an arbitrary edge of \mathcal{C}_3 . If $n \geq 7$, then for every vertices $z \in W$ and $v \in (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\}) \cup (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\})$ the edge $(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, v\}$ is red. If $n = 5$, then for every vertices $z \in W$ and $v \in \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}$ the edge*

$$(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, v\}$$

is red.

Proof of Claim 4.5. We give only a proof for $n \geq 7$. The proof for $n = 5$ is similar. Suppose indirectly that there are vertices $z \in W$ and $v \in (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\}) \cup (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\})$ so that the edge

$$g = (e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, v\}$$

is blue. With no loss of generality assume that $v \in e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}$. Using Claim 4.2 with $f_j = f_1$, there are vertices $\bar{u} \in f_{\frac{n+1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}\}$, $\bar{u}' \in f_2 \setminus \{l_{\mathcal{C}_2, f_2}\}$ and $\hat{u} \in f_1 \setminus \{f_{\mathcal{C}_2, f_1}, l_{\mathcal{C}_2, f_1}\}$ so that the edge

$$g' = (f_1 \setminus \{f_{\mathcal{C}_2, f_1}, l_{\mathcal{C}_2, f_1}, \hat{u}\}) \cup \{\bar{u}, \bar{u}', z\}$$

is blue. Use Claim 4.3 to obtain a blue edge

$$g'' = (f_{\frac{n+1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n+1}{2}}}, \bar{u}\}) \cup \{\bar{x}, \bar{y}\}$$

for some $\bar{x} \in e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}, l_{\mathcal{C}_3, e_{i-1}}\}$ and $\bar{y} \in f_{\frac{n-1}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-1}{2}}}\}$. Clearly

$$e_1 e_2 \dots e_{i-1} g'' f_{\frac{n-1}{2}} f_{\frac{n-3}{2}} \dots f_2 g' g e_{i+1} e_{i+2} \dots e_{\frac{n-1}{2}}$$

is a blue copy of \mathcal{C}_n^k , a contradiction to our assumption. This contradiction completes the proof of Claim 4.5. \square

Now, set

$$h_1 = \begin{cases} (e_1 \setminus \{v_k\}) \cup \{z_1\} & n \equiv 1, \\ (e_1 \setminus \{v_1, v_k\}) \cup \{z_1, v_{(k-1)(\frac{n-1}{2})}\} & n \equiv 3, \end{cases}$$

and for $2 \leq i \leq \frac{n-1}{2}$,

$$h_i = \begin{cases} (e_i \setminus \{l_{\mathcal{C}_3, e_i}\}) \cup \{z_{\frac{i+1}{2}}\} & \text{if } i \text{ is odd,} \\ (e_i \setminus \{f_{\mathcal{C}_3, e_i}\}) \cup \{z_{\frac{i}{2}}\} & \text{if } i \text{ is even,} \end{cases}$$

where $\{z_1, z_2, \dots, z_{\frac{n-1}{4}}\} \subseteq V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_3)$. Using Claims 4.4 and 4.5, $\mathcal{C}_4 = h_1 h_2 \dots h_{\frac{n-1}{2}}$ is a red copy of $\mathcal{C}_{\frac{n-1}{2}}^k$ disjoint from \mathcal{C}_1 . The proof of the following claim is similar to the proof of Claim 4.3. So we omit it here.

Claim 4.6 *Let d_i and h_j be two arbitrary edges of \mathcal{C}_1 and \mathcal{C}_4 , respectively. For every vertex $\bar{u}' \in d_i \setminus \{f_{\mathcal{C}_1, d_i}\}$ (resp. $\bar{u}' \in d_i \setminus \{l_{\mathcal{C}_1, d_i}\}$), there are vertices $\bar{v} \in h_j \setminus \{f_{\mathcal{C}_4, h_j}, l_{\mathcal{C}_4, h_j}\}$ and $\bar{u} \in d_{i-1} \setminus \{f_{\mathcal{C}_1, d_{i-1}}\}$ (resp. $\bar{u} \in d_{i+1} \setminus \{l_{\mathcal{C}_1, d_{i+1}}\}$) so that the edge $V(d_i) \setminus \{f_{\mathcal{C}_1, d_i}, \bar{u}'\} \cup \{\bar{v}, \bar{u}\}$ (resp. the edge $(V(d_i) \setminus \{l_{\mathcal{C}_1, d_i}, \bar{u}'\}) \cup \{\bar{v}, \bar{u}\})$ is red.*

Now, using Claim 4.6 there are vertices $\bar{v} \in h_1 \setminus \{f_{\mathcal{C}_4, h_1}, l_{\mathcal{C}_4, h_1}\}$ and $\bar{u} \in d_{\frac{n+1}{2}} \setminus \{f_{\mathcal{C}_1, d_{\frac{n+1}{2}}}\}$ so that the edge

$$g = (V(d_1) \setminus \{f_{\mathcal{C}_1, d_1}, l_{\mathcal{C}_1, d_1}\}) \cup \{\bar{u}, \bar{v}\}$$

is red (by putting $i = j = 1$ and $\overline{u'} = l_{\mathcal{C}_1, d_1}$). Now let $z = l_{\mathcal{C}_1, d_1}$. If $n \neq 5$, then set $y \in (h_3 \setminus \{f_{\mathcal{C}_4, h_3}, l_{\mathcal{C}_4, h_3}\}) \cap (e_3 \setminus \{f_{\mathcal{C}_3, e_3}, l_{\mathcal{C}_3, e_3}\})$ and if $n = 5$, then set $y = f_{\mathcal{C}_3, e_1}$. When $z \in \mathcal{C}_2$, Claim 4.4 and when $z \in W$, Claim 4.5 implies that the edge

$$g' = (h_2 \setminus \{f_{\mathcal{C}_4, h_2}, l_{\mathcal{C}_4, h_2}\}) \cup \{z, y\}$$

is red. Now, clearly

$$gd_{\frac{n+1}{2}}d_{\frac{n-1}{2}} \dots d_2g'h_3h_4 \dots h_{\frac{n-1}{2}}h_1$$

for $n \neq 5$ and $gd_3d_2g'h_1$ for $n = 5$ is a red copy of \mathcal{C}_n^k , a contradiction.

Case 2: $n \equiv 0 \pmod{4}$

In this case, we show that there are two disjoint isochromatic $\mathcal{C}_{\frac{n}{2}}^k$. Since

$$R(\mathcal{C}_{\frac{n}{2}}^k, \mathcal{C}_{\frac{n}{2}}^k) = f(\frac{n}{2}) < f(n),$$

there is a monochromatic $\mathcal{C}_1 = \mathcal{C}_{\frac{n}{2}}^k$. By symmetry we may assume that $\mathcal{C}_1 \subseteq \mathcal{H}_{blue}$. Since $|V(\mathcal{C}_1)| = (k-1)(\frac{n}{2})$ and

$$f(\frac{n}{2}) < f(n) - (k-1)(\frac{n}{2}),$$

there is a monochromatic $\mathcal{C}_2 = \mathcal{C}_{\frac{n}{2}}^k$ in $V(\mathcal{H}) \setminus V(\mathcal{C}_1)$. If \mathcal{C}_2 is blue, we are done. So suppose that \mathcal{C}_2 is red. Among all red-blue copies of $\mathcal{C}_{\frac{n}{2}}^k$'s, choose red-blue copies with maximum intersection, say \mathcal{C}'_1 and \mathcal{C}'_2 . Similar to the proof of Claim 4.1 we have

$$|V(\mathcal{C}'_1) \cup V(\mathcal{C}'_2)| \leq R(\mathcal{C}_{\frac{n}{2}}^k, \mathcal{C}_{\frac{n}{2}}^k) + 1 = f(\frac{n}{2}) + 1.$$

Since

$$f(n) - |V(\mathcal{C}'_1) \cup V(\mathcal{C}'_2)| \geq f(\frac{n}{2}) = R(\mathcal{C}_{\frac{n}{2}}^k, \mathcal{C}_{\frac{n}{2}}^k),$$

there is a monochromatic $\mathcal{C}_{\frac{n}{2}}^k$, say \mathcal{C} , disjoint from \mathcal{C}'_1 and \mathcal{C}'_2 . Clearly \mathcal{C} with one of \mathcal{C}'_1 and \mathcal{C}'_2 form our favorable cycles.

With no loss of generality assume that \mathcal{C}_1 and \mathcal{C}_2 are two blue $\mathcal{C}_{\frac{n}{2}}^k$. Let $\mathcal{C}_1 = e_1e_2 \dots e_{\frac{n}{2}}$, $\mathcal{C}_2 = f_1f_2 \dots f_{\frac{n}{2}}$ and $W = V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)$ where

$$e_i = \{v_1, v_2, \dots, v_k\} + (k-1)(i-1)(\text{mod } (k-1)\frac{n}{2}), \quad i = 1, 2, \dots, \frac{n}{2}$$

and

$$f_i = \{u_1, u_2, \dots, u_k\} + (k-1)(i-1)(\text{mod } (k-1)\frac{n}{2}), \quad i = 1, 2, \dots, \frac{n}{2}.$$

Since $n \geq 8$, we have $|W| \geq 3$. Use Lemma 3.8 for $e_i = e_1$, $f_j = f_1$ and $B = \{w_1, w_2, w_3\} \subseteq W$ to obtain two red paths E_1 and F_1 with desired properties in Lemma 3.8. As mentioned in Lemma 3.8, there are distinct vertices $\overline{v} \in e_1 \setminus (V(E_1) \cup$

$V(F_1))$, $v \in e_1 \setminus (V(E_1) \cup \{v_1, v_k, \bar{v}\})$, $\bar{u} \in f_1 \setminus (V(E_1) \cup V(F_1))$ and $u \in f_1 \setminus (V(F_1) \cup \{u_1, u_k, \bar{u}\})$. If $\bar{v} \neq v_1$, then set $g_i = e_i$ for $1 \leq i \leq \frac{n}{2}$. If $\bar{v} = v_1$ then set $g_1 = e_1$ and $g_i = e_{\frac{n}{2}-i+2}$ for $2 \leq i \leq \frac{n}{2}$. If $\bar{u} \neq u_1$, then set $h_i = f_i$ for $1 \leq i \leq \frac{n}{2}$. If $\bar{u} = u_1$ then set $h_1 = f_1$ and $h_i = f_{\frac{n}{2}-i+2}$ for $2 \leq i \leq \frac{n}{2}$. Clearly $\mathcal{C}_1 = g_1 g_2 \dots g_{\frac{n}{2}}$ and $\mathcal{C}_2 = h_1 h_2 \dots h_{\frac{n}{2}}$. Assume that

$$g_i = \{x_1, x_2, \dots, x_k\} + (k-1)(i-1) \pmod{(k-1)\frac{n}{2}}, \quad i = 1, 2, \dots, \frac{n}{2},$$

$$h_i = \{y_1, y_2, \dots, y_k\} + (k-1)(i-1) \pmod{(k-1)\frac{n}{2}}, \quad i = 1, 2, \dots, \frac{n}{2}.$$

Now, use Lemma 3.9 for $e_i = g_2$, $f_j = h_2$ and $\mathcal{E}_1 = E_1$ (resp. $\mathcal{E}_1 = F_1$) to obtain two red paths \mathcal{E}_2 and \mathcal{F}_2 (resp. $\bar{\mathcal{E}}_2$ and $\bar{\mathcal{F}}_2$) of length 4 with desired properties of Lemma 3.9. Apply Lemma 3.10 for \mathcal{E}_l , \mathcal{F}_l (resp. for $\bar{\mathcal{E}}_l$ and $\bar{\mathcal{F}}_l$), $e_i = g_{l+1}$ and $f_j = h_{l+1}$ where $2 \leq l \leq \frac{n}{2} - 2$ to obtain two red paths \mathcal{E}_{l+1} and \mathcal{F}_{l+1} (resp. $\bar{\mathcal{E}}_{l+1}$ and $\bar{\mathcal{F}}_{l+1}$) with properties of Lemma 3.10. Let

$$\mathcal{E}_{\frac{n}{2}-1} = p_1 \bar{p}_2 \dots p_{n-2}, \mathcal{F}_{\frac{n}{2}-1} = p_1 p_2 p'_3 \dots p'_{n-2}, \bar{\mathcal{E}}_{\frac{n}{2}-1} = p_1 \bar{p}_2 \dots \bar{p}_{n-2}$$

and $\bar{\mathcal{F}}_{\frac{n}{2}-1} = p_1 \bar{p}_2 \bar{p}'_3 \dots \bar{p}'_{n-2}$. Also, let $x \in g_{\frac{n}{2}-1} \setminus (\mathcal{E}_{\frac{n}{2}-1} \cup \{f_{\mathcal{C}_1, g_{\frac{n}{2}-1}}\})$, $y \in h_{\frac{n}{2}-1} \setminus (\mathcal{F}_{\frac{n}{2}-1} \cup \{f_{\mathcal{C}_2, h_{\frac{n}{2}-1}}\})$, $\bar{x} \in g_{\frac{n}{2}-1} \setminus (\bar{\mathcal{E}}_{\frac{n}{2}-1} \cup \{f_{\mathcal{C}_1, g_{\frac{n}{2}-1}}\})$, $\bar{y} \in h_{\frac{n}{2}-1} \setminus (\bar{\mathcal{F}}_{\frac{n}{2}-1} \cup \{f_{\mathcal{C}_2, h_{\frac{n}{2}-1}}\})$, $y'' \in (h_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_2, h_{\frac{n}{2}-1}}\}) \cap (p_{n-2} \setminus p_{n-3})$ and $x'' \in (g_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_1, g_{\frac{n}{2}-1}}\}) \cap (\bar{p}'_{n-2} \setminus \bar{p}'_{n-3})$. Consider an edge $q = E \cup F$ in $\mathcal{A}_{\frac{n}{2}, \frac{n}{2}}$ so that

$$E = \{x, x_{(k-1)(\frac{n}{2}-1)+2}, \dots, x_{(k-1)(\frac{n}{2}-1)+\lfloor \frac{k}{2} \rfloor}\},$$

$$F = \{y_{(k-1)\frac{n}{2}-\lceil \frac{k}{2} \rceil+2}, \dots, y_{(k-1)\frac{n}{2}}, y'\},$$

where $y' \in (h_1 \setminus \{y_k\}) \cap (p_1 \setminus p_2)$.

Claim 4.7 *The edge q is red.*

Proof of Claim 4.7. Suppose indirectly that the edge $q_1 = q$ is blue. Since there is no blue copy of \mathcal{C}_n^k , using Remark 3.5, every edge in $\mathcal{B}_{\frac{n}{2}, \frac{n}{2}}$ that is disjoint from q_1 is red. For $2 \leq l \leq \lfloor \frac{k}{2} \rfloor$, let $q_l = (q_{l-1} \setminus \{x_{(k-1)(\frac{n}{2}-1)+l}\}) \cup \{x_{(k-1)\frac{n}{2}-l+2}\}$. Assume that l' is the maximum $l \in [1, \lfloor \frac{k}{2} \rfloor]$ for which q_l is blue. If $l' < \lfloor \frac{k}{2} \rfloor$, then $q_{l'+1}$ is red. Set

$$q'_{l'+1} = \left((g_{\frac{n}{2}} \cup h_{\frac{n}{2}}) \setminus (q_{l'} \cup \{f_{\mathcal{C}_1, g_{\frac{n}{2}}}, l_{\mathcal{C}_1, g_{\frac{n}{2}}}, f_{\mathcal{C}_2, h_{\frac{n}{2}}}, l_{\mathcal{C}_2, h_{\frac{n}{2}}}\}) \right) \cup \{v, y''\}.$$

Clearly, $q'_{l'+1}$ is an edge in $\mathcal{B}_{\frac{n}{2}, \frac{n}{2}}$ disjoint from $q_{l'}$. Since there is no blue copy of \mathcal{C}_n^k , the edge $q'_{l'+1}$ is red. Therefore, $\mathcal{E}_{\frac{n}{2}-1} q'_{l'+1} q_{l'+1}$ is a red copy of \mathcal{C}_n^k , a contradiction. Therefore, we may assume that $l' = \lfloor \frac{k}{2} \rfloor$ and hence the edge $q_{\lfloor \frac{k}{2} \rfloor} = E' \cup F$ is blue, where

$$E' = \{x, x_{(k-1)\frac{n}{2}-\lfloor \frac{k}{2} \rfloor+2}, \dots, x_{(k-1)\frac{n}{2}-1}, x_{(k-1)\frac{n}{2}}\}.$$

Now, set $m = \lfloor \frac{k}{2} \rfloor - 1$. For $1 \leq l \leq m$, let

$$q_{\lfloor \frac{k}{2} \rfloor + l} = (q_{\lfloor \frac{k}{2} \rfloor + l - 1} \setminus \{y_{(k-1)\frac{n}{2}-l+1}\}) \cup \{y_{(k-1)(\frac{n}{2}-1)+l+1}\}.$$

Now, let l' be the maximum $l \in [0, m]$ for which $q_{\lfloor \frac{k}{2} \rfloor + l}$ is blue. If $l' < m$, then $q_{\lfloor \frac{k}{2} \rfloor + l' + 1}$ is red. Set

$$q'_{\lfloor \frac{k}{2} \rfloor + l' + 1} = \left((g_{\frac{n}{2}} \cup h_{\frac{n}{2}}) \setminus (q_{\lfloor \frac{k}{2} \rfloor + l'} \cup \{fc_{1, g_{\frac{n}{2}}}, lc_{1, g_{\frac{n}{2}}}, fc_{2, h_{\frac{n}{2}}}, lc_{2, h_{\frac{n}{2}}}\}) \right) \cup \{v, y''\}.$$

Since there is no blue copy of \mathcal{C}_n^k , the edge $q'_{\lfloor \frac{k}{2} \rfloor + l' + 1}$ is red. So

$$\mathcal{E}_{\frac{n}{2}-1} q'_{\lfloor \frac{k}{2} \rfloor + l' + 1} q_{\lfloor \frac{k}{2} \rfloor + l' + 1}$$

is a red copy of \mathcal{C}_n^k , a contradiction. So we may assume that $l' = m$ and hence the edge $q_{\lfloor \frac{k}{2} \rfloor + m} = E' \cup F'$ is blue, where

$$F' = \begin{cases} \{y_{(k-1)(\frac{n}{2}-1)+2}, \dots, y_{(k-1)(\frac{n}{2}-1)+m+1}, y'\} & \text{if } k \text{ is even,} \\ \{y_{(k-1)(\frac{n}{2}-1)+2}, \dots, y_{(k-1)(\frac{n}{2}-1)+m+1}, y_{(k-1)(\frac{n}{2}-1)+m+2}, y'\} & \text{if } k \text{ is odd.} \end{cases}$$

Let

$$q_{\lfloor \frac{k}{2} \rfloor + m + 1} = (q_{\lfloor \frac{k}{2} \rfloor + m} \setminus \{x\}) \cup \{v\}.$$

If $q_{\lfloor \frac{k}{2} \rfloor + m + 1}$ is red, then set

$$q'_{\lfloor \frac{k}{2} \rfloor + m + 1} = \left((g_{\frac{n}{2}} \cup h_{\frac{n}{2}}) \setminus (q_{\lfloor \frac{k}{2} \rfloor + m} \cup \{fc_{1, g_{\frac{n}{2}}}, lc_{1, g_{\frac{n}{2}}}, fc_{2, h_{\frac{n}{2}}}, lc_{2, h_{\frac{n}{2}}}\}) \right) \cup \{v, y''\}.$$

Since there is no blue copy of \mathcal{C}_n^k , the edge $q'_{\lfloor \frac{k}{2} \rfloor + m + 1}$ is red and

$$\mathcal{E}_{\frac{n}{2}-1} q'_{\lfloor \frac{k}{2} \rfloor + m + 1} q_{\lfloor \frac{k}{2} \rfloor + m + 1}$$

is a red copy of \mathcal{C}_n^k , a contradiction. So we may assume that the edge $q_{\lfloor \frac{k}{2} \rfloor + m + 1}$ is blue.

Now, let

$$q_{\lfloor \frac{k}{2} \rfloor + m + 2} = (q_{\lfloor \frac{k}{2} \rfloor + m + 1} \setminus \{y', v\}) \cup \{\overline{y}, x'\}$$

where $x' \in (g_1 \setminus \{x_k\}) \cap (p_1 \setminus \overline{p}_2)$.

If $q_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is red, then set

$$q'_{\lfloor \frac{k}{2} \rfloor + m + 2} = \left((g_{\frac{n}{2}} \cup h_{\frac{n}{2}}) \setminus (q_{\lfloor \frac{k}{2} \rfloor + m + 1} \cup \{fc_{1, g_{\frac{n}{2}}}, lc_{1, g_{\frac{n}{2}}}, fc_{2, h_{\frac{n}{2}}}, lc_{2, h_{\frac{n}{2}}}\}) \right) \cup \{\overline{y}, x''\}.$$

Since there is no blue copy of \mathcal{C}_n^k , the edge $q'_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is red and

$$\overline{\mathcal{F}}_{\frac{n}{2}-1} q'_{\lfloor \frac{k}{2} \rfloor + m + 2} q_{\lfloor \frac{k}{2} \rfloor + m + 2}$$

is a red copy of \mathcal{C}_n^k , a contradiction. So we may assume that the edge $q_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is blue.

If k is even, then clearly $q_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is an edge in $\mathcal{B}_{\frac{n}{2} \frac{n}{2}}$ disjoint from q_1 . This is impossible, by Remark 3.5. Now we may assume that k is odd. One can easily see that $x_{(k-1)(\frac{n}{2}-1) + \frac{k+1}{2}} \notin q_1 \cup q_{\lfloor \frac{k}{2} \rfloor + m + 2}$ and $y_{(k-1)(\frac{n}{2}-1) + \frac{k+1}{2}} \in q_1 \cap q_{\lfloor \frac{k}{2} \rfloor + m + 2}$. Similarly, we can show that the edge

$$q_{\lfloor \frac{k}{2} \rfloor + m + 3} = q_{\lfloor \frac{k}{2} \rfloor + m + 2} \setminus \{y_{(k-1)(\frac{n}{2}-1) + \frac{k+1}{2}}\} \cup \{x_{(k-1)(\frac{n}{2}-1) + \frac{k+1}{2}}\}$$

is blue. That is a contradiction to Remark 3.5. This contradiction completes the proof of Claim 4.7. \square

Now, consider an edge $q' = E' \cup F'$ in $\mathcal{B}_{\frac{n}{2} \frac{n}{2}}$ so that

$$\begin{aligned} E' &= \{x_{(k-1)(\frac{n}{2}-1) + \lfloor \frac{k}{2} \rfloor}, x_{(k-1)(\frac{n}{2}-1) + \lfloor \frac{k}{2} \rfloor + 2}, x_{(k-1)(\frac{n}{2}-1) + \lfloor \frac{k}{2} \rfloor + 3}, \dots, x_{(k-1)\frac{n}{2}}, v\}, \\ F' &= \{y'', y_{(k-1)(\frac{n}{2}-1) + 2}, \dots, y_{(k-1)(\frac{n}{2}-1) + \lfloor \frac{k}{2} \rfloor}\}. \end{aligned}$$

By an argument similar to the proof of Claim 4.7 we can show that q' is red. Clearly $\mathcal{E}_{\frac{n}{2}-1} q' q$ is a red copy of \mathcal{C}_n^k , a contradiction to our assumption.

Case 3: $n \equiv 2 \pmod{4}$

By an argument similar to the proof of Claim 4.1 we have the following claim.

Claim 4.8 *There are two monochromatic copies of $\mathcal{C}_{\frac{n}{2}+1}^k$ of colors red and blue.*

Among all red-blue copies of $\mathcal{C}_{\frac{n}{2}+1}^k$'s choose red-blue copies with maximum intersection. Let $\mathcal{C}_1 = d_1 d_2 \dots d_{\frac{n}{2}+1} \subseteq \mathcal{H}_{\text{red}}$ and $\mathcal{C}_2 \subseteq \mathcal{H}_{\text{blue}}$ be such copies. It is easy to see that $|V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq R(\mathcal{C}_{\frac{n}{2}+1}^k, \mathcal{C}_{\frac{n}{2}+1}^k) + 1$. Since

$$|V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)| \geq f\left(\frac{n}{2} - 1\right),$$

using induction hypothesis, there is a monochromatic $\mathcal{C}_{\frac{n}{2}-1}^k$ in $V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)$, say \mathcal{C}_3 . By symmetry we may assume that $\mathcal{C}_3 \subseteq \mathcal{H}_{\text{blue}}$. Let $\mathcal{C}_2 = f_1 f_2 \dots f_{\frac{n}{2}+1}$, $\mathcal{C}_3 = e_1 e_2 \dots e_{\frac{n}{2}-1}$ and $W = V(\mathcal{H}) \setminus V(\mathcal{C}_2 \cup \mathcal{C}_3)$ where

$$f_i = \{u_1, u_2, \dots, u_k\} + (k-1)(i-1) \pmod{(k-1)\left(\frac{n}{2} + 1\right)}, \quad i = 1, 2, \dots, \frac{n}{2} + 1$$

and

$$e_i = \{v_1, v_2, \dots, v_k\} + (k-1)(i-1) \pmod{(k-1)\left(\frac{n}{2} - 1\right)}, \quad i = 1, 2, \dots, \frac{n}{2} - 1.$$

We have the following Claims.

Claim 4.9 *Let e_i and f_j be two arbitrary edges of \mathcal{C}_3 and \mathcal{C}_2 , respectively and $y \in e_i$. For every vertices $x \in f_j \setminus \{f_{\mathcal{C}_2, f_j}\}$ and $x' \in f_{j+2} \setminus \{l_{\mathcal{C}_2, f_{j+2}}\}$, there are distinct vertices $v, v' \in e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}, y\}$, $u \in f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}$ and $u' \in f_{j+3} \setminus \{l_{\mathcal{C}_2, f_{j+3}}\}$ so that at least one of the edges $g = (V(f_j) \setminus \{f_{\mathcal{C}_2, f_j}, x\}) \cup \{v, u\}$ or*

$$g' = (V(f_{j+2}) \setminus \{x', l_{\mathcal{C}_2, f_{j+2}}\}) \cup \{v', u'\}$$

is blue.

Proof of Claim 4.9. Suppose to the contrary that there are vertices $x \in f_j \setminus \{fc_{2,f_j}\}$ and $x' \in f_{j+2} \setminus \{lc_{2,f_{j+2}}\}$ so that for every distinct vertices $v, v' \in e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}, y\}$, $u \in f_{j-1} \setminus \{fc_{2,f_{j-1}}\}$ and $u' \in f_{j+3} \setminus \{lc_{2,f_{j+3}}\}$ the edges

$$g = (V(f_j) \setminus \{fc_{2,f_j}, x\}) \cup \{v, u\}$$

and

$$g' = (V(f_{j+2}) \setminus \{x', lc_{2,f_{j+2}}\}) \cup \{v', u'\}$$

are red. By symmetry we may assume that $e_i = e_{\frac{n}{2}-1}$ and $f_j = f_{\frac{n}{2}-1}$.

Use Lemma 3.6 for $e = e_1$ (resp. $e = f_1$) to obtain a red path $\mathcal{E}_1 = g_1 g'_1$ (resp. $\mathcal{F}_1 = \overline{g}_1 \overline{g}'_1$) with the mentioned properties of Lemma 3.6 (by putting $i = j = 1$, $v' = v_1$, $v'' = v_k$, $u' = u_1$, $u'' = u_k$, $C = \{v_{k-1}\}$ and $B = \{w_1, w_2\} \subseteq W$). Use Lemma 3.10, $\frac{n}{2} - 3$ times, to obtain two red paths $\mathcal{E}_{\frac{n}{2}-2}$ and $\mathcal{F}_{\frac{n}{2}-2}$ of length $n - 4$ with the properties of Lemma 3.10.

With no loss of generality assume that $y' \in e_{\frac{n}{2}-2} \setminus (\mathcal{E}_{\frac{n}{2}-2} \cup \{fc_{3,e_{\frac{n}{2}-2}}, v_{k-1}\})$. Use Lemma 3.7 for $i = \frac{n}{2} - 1$, $j = \frac{n}{2}$, $v' = y'$, $v'' = v_{k-1}$, $u' = x$, $u'' = x'$ to obtain a red path $\mathcal{P} = g_{\frac{n}{2}-1} g'_{\frac{n}{2}-1}$ with the properties of Lemma 3.7.

Let $\mathcal{E}_{\frac{n}{2}-2} = h_1 h_2 \dots h_{n-4}$, $\mathcal{P} = h_{n-3} h_{n-2}$, $v \in (e_{\frac{n}{2}-1} \setminus \{fc_{3,e_{\frac{n}{2}-1}}, v_1, y\}) \cap (h_{n-3} \setminus h_{n-2})$, $v' \in (e_{\frac{n}{2}-1} \setminus \{fc_{3,e_{\frac{n}{2}-1}}, v_1, y\}) \cap (h_{n-2} \setminus h_{n-3})$, $u \in (f_{\frac{n}{2}-2} \setminus \{fc_{2,f_{\frac{n}{2}-2}}\}) \cap (h_{n-4} \setminus h_{n-5})$ and $u' \in (f_1 \setminus \{lc_{2,f_1}\}) \cap (h_1 \setminus h_2)$. Clearly, $\mathcal{E}_{\frac{n}{2}-2} g \mathcal{P} g'$ is a red copy of \mathcal{C}_n^k . This contradiction completes the proof of Claim 4.9. \square

Claim 4.10 *Let e_i and f_j be two arbitrary edges of \mathcal{C}_3 and \mathcal{C}_2 , respectively. Also, let $A = (e_{i+1} \setminus \{lc_{3,e_{i+1}}\}) \cup (e_{i-1} \setminus \{fc_{3,e_{i-1}}\})$. For every vertices $z \in f_j \setminus \{fc_{2,f_j}, lc_{2,f_j}\}$ and $\overline{v} \in A$ the edge $(e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}\}) \cup \{z, \overline{v}\}$ is red.*

Proof of Claim 4.10. By symmetry it only suffices to show that for every vertices $z \in f_j \setminus \{fc_{2,f_j}, lc_{2,f_j}\}$ and $\overline{v} \in e_{i+1} \setminus \{lc_{3,e_{i+1}}\}$, the edge

$$(e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}\}) \cup \{z, \overline{v}\}$$

is red. With no loss of generality we may assume that $e_i = e_1$ and $f_j = f_{\frac{n}{2}}$. Suppose for the sake of contradiction that there are vertices $z \in f_{\frac{n}{2}} \setminus \{fc_{2,f_{\frac{n}{2}}}, lc_{2,f_{\frac{n}{2}}}\}$ and $\overline{v} \in e_2 \setminus \{lc_{3,e_2}\}$ so that the edge

$$h = (e_1 \setminus \{v_1, v_k\}) \cup \{z, \overline{v}\}$$

is blue. Since there is no blue copy of \mathcal{C}_n^k , then for every distinct vertices $v, v' \in e_{\frac{n}{2}-1} \setminus \{fc_{3,e_{\frac{n}{2}-1}}, lc_{3,e_{\frac{n}{2}-1}}, \overline{v}\}$, $u \in f_{\frac{n}{2}-2} \setminus \{fc_{2,f_{\frac{n}{2}-2}}\}$ and $u' \in f_1 \setminus \{lc_{2,f_1}\}$ the edges

$$g = (f_{\frac{n}{2}-1} \setminus \{fc_{2,f_{\frac{n}{2}-1}}, lc_{2,f_{\frac{n}{2}-1}}\}) \cup \{v, u\}$$

and

$$g' = (f_{\frac{n}{2}+1} \setminus \{fc_{2,f_{\frac{n}{2}+1}}, lc_{2,f_{\frac{n}{2}+1}}\}) \cup \{v', u'\}$$

are red. That is a contradiction to Claim 4.9. \square

Claim 4.11 Let e_i and f_j be two arbitrary edges of \mathcal{C}_3 and \mathcal{C}_2 , respectively and $\bar{u} \in \{fc_{2,f_j}, lc_{2,f_j}\}$. For $n \geq 10$ and $\bar{v} \in e_{i+1} \setminus \{lc_{3,e_{i+1}}\}$ (also for $n = 6$ and $\bar{v} = lc_{3,e_i}$), if the edge

$$(e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}\}) \cup \{\bar{u}, \bar{v}\}$$

is blue, then for every vertices $\hat{v} \in e_{i+1} \setminus \{lc_{3,e_{i+1}}\}$ and $\hat{u} \in \{fc_{2,f_j}, lc_{2,f_j}\} \setminus \{\bar{u}\}$ the edge $(e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}\}) \cup \{\hat{u}, \hat{v}\}$ is red. For $n \geq 10$ and $\bar{v} \in e_{i-1} \setminus \{fc_{3,e_{i-1}}\}$ (also for $n = 6$ and $\bar{v} = fc_{3,e_i}$), if the edge

$$(e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}\}) \cup \{\bar{u}, \bar{v}\}$$

is blue, then for every vertices $\hat{v} \in e_{i-1} \setminus \{fc_{3,e_{i-1}}\}$ and $\hat{u} \in \{fc_{2,f_j}, lc_{2,f_j}\} \setminus \{\bar{u}\}$ the edge $(e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}\}) \cup \{\hat{u}, \hat{v}\}$ is red.

Proof of Claim 4.11. We give only a proof for $n \geq 10$. The proof for $n = 6$ is similar. By symmetry we may assume that $\bar{v} \in e_{i+1} \setminus \{lc_{3,e_{i+1}}\}$ and $\bar{u} = fc_{2,f_j}$. With no loss of generality we may assume that $e_i = e_1$ and $f_j = f_{\frac{n}{2}}$. Assume for the sake of contradiction that there is a vertex $\hat{v} \in e_2 \setminus \{lc_{3,e_2}\}$ so that the edge

$$(e_1 \setminus \{v_1, v_k\}) \cup \{lc_{2,f_{\frac{n}{2}}}, \hat{v}\}$$

is blue. Since there is no blue copy of \mathcal{C}_n^k , then for every distinct vertices $v, v' \in e_{\frac{n}{2}-1} \setminus \{fc_{3,e_{\frac{n}{2}-1}}, lc_{3,e_{\frac{n}{2}-1}}\}$, $u \in f_{\frac{n}{2}-2} \setminus \{fc_{2,f_{\frac{n}{2}-2}}\}$ and $u' \in f_1 \setminus \{lc_{2,f_1}\}$ the edges

$$g = (f_{\frac{n}{2}-1} \setminus \{fc_{2,f_{\frac{n}{2}-1}}, lc_{2,f_{\frac{n}{2}-1}}\}) \cup \{v, u\}$$

and

$$g' = (f_{\frac{n}{2}+1} \setminus \{fc_{2,f_{\frac{n}{2}+1}}, lc_{2,f_{\frac{n}{2}+1}}\}) \cup \{v', u'\}$$

are red, a contradiction to Claim 4.9. \square

Claim 4.12 Let e_i be an arbitrary edge of \mathcal{C}_3 , $z \in W$ and $\bar{v} \in (e_{i+1} \setminus \{lc_{3,e_{i+1}}\}) \cup (e_{i-1} \setminus \{fc_{3,e_{i-1}}\})$. If the edge

$$h = (e_i \setminus \{fc_{3,e_i}, lc_{3,e_i}\}) \cup \{z, \bar{v}\}$$

is blue, then for every edge $f_j \in \mathcal{C}_2$ and every distinct vertices $x \in f_{j-1} \setminus \{fc_{2,f_{j-1}}\}$, $x' \in f_{j+1} \setminus \{lc_{2,f_{j+1}}\}$ and $\hat{u} \in f_j \setminus \{fc_{2,f_j}, lc_{2,f_j}\}$ the edge

$$h' = (f_j \setminus \{fc_{2,f_j}, lc_{2,f_j}, \hat{u}\}) \cup \{x, x', z\}$$

is red.

Proof of Claim 4.12. By symmetry we may assume that $\bar{v} \in e_{i+1} \setminus \{lc_{3,e_{i+1}}\}$. Suppose to the contrary that there is an edge f_j and there are distinct vertices $x \in f_{j-1} \setminus \{fc_{2,f_{j-1}}\}$, $x' \in f_{j+1} \setminus \{lc_{2,f_{j+1}}\}$ and $\hat{u} \in f_j \setminus \{fc_{2,f_j}, lc_{2,f_j}\}$ so that the edge

$$h' = (f_j \setminus \{fc_{2,f_j}, lc_{2,f_j}, \hat{u}\}) \cup \{x, x', z\}$$

is blue. Since there is no blue copy of \mathcal{C}_n^k , then for every distinct vertices $v, v' \in e_{i-1} \setminus \{fc_{3,e_{i-1}}, \bar{v}\}$, $u \in f_{j-2} \setminus \{fc_{2,f_{j-2}}\}$ and $u' \in f_{j+2} \setminus \{lc_{2,f_{j+2}}\}$ the edges $(f_{j-1} \setminus \{fc_{2,f_{j-1}}, x\}) \cup \{v, u\}$ and $(f_{j+1} \setminus \{x', lc_{2,f_{j+1}}\}) \cup \{v', u'\}$ are red, a contradiction to Claim 4.9. So we are done. \square

Claim 4.13 Let $n = 6$, $W = \{z_1, z_2\}$ and e_i be an arbitrary edge of \mathcal{C}_3 and $v \in \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}$. If the edge

$$(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z_1, v\}$$

is blue, then for every vertex $\bar{v} \in (e_{i+1} \setminus \{f_{\mathcal{C}_3, e_{i+1}}, l_{\mathcal{C}_3, e_{i+1}}\}) \cup \{v\}$ the edge

$$(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z_2, \bar{v}\}$$

is red.

Proof of Claim 4.13. By symmetry we may assume that $e_i = e_1$ and $v = v_k$. Suppose for the sake of contradiction that there is a vertex $\bar{v} \in e_2 \setminus \{v_1\}$ so that the edges $(e_1 \setminus \{v_1\}) \cup \{z_1\}$ and $(e_1 \setminus \{v_1, v_k\}) \cup \{z_2, \bar{v}\}$ are blue. Let $u \in f_4 \setminus \{f_{\mathcal{C}_2, f_4}, l_{\mathcal{C}_2, f_4}\}$ and $\bar{u} \in f_2 \setminus \{f_{\mathcal{C}_2, f_2}, l_{\mathcal{C}_2, f_2}\}$. Use Lemma 3.6 for $e = e_1$ to obtain a red path $\mathcal{P} = g_1 g'_1$ with the mentioned properties of Lemma 3.6 (by putting $i = j = 1$, $v' = v_1$, $v'' = v_k$, $u' = u$, $u'' = \bar{u}$, $C = \{v_{k-1}\}$ and $B = W = \{z_1, z_2\}$). By Lemma 3.6, there is a vertex $w \in e_1 \setminus \{v_1, v_{k-1}\}$ so that $w \notin \mathcal{P}$. Now, use Lemma 3.7 for $e_i = e_2$, $f_j = f_3$, $v' = w$, $v'' = v_{k-1}$, $u' = f_{\mathcal{C}_2, f_3}$ and $u'' = l_{\mathcal{C}_2, f_3}$ to obtain a red path $\mathcal{P}' = g_2 g'_2$ with the mentioned properties of Lemma 3.7. Let $x \in (f_3 \setminus \{l_{\mathcal{C}_2, f_3}\}) \cap (g_2 \setminus g'_2)$, $x' \in (f_3 \setminus \{f_{\mathcal{C}_2, f_3}\}) \cap (g'_2 \setminus g_2)$ and $y \in (f_1 \setminus \{u_1, u_k\}) \cap (g'_1 \setminus g_1)$. As mentioned in Lemma 3.6, we may assume that $z_1 \in g_1$. Now, let

$$h = (f_4 \setminus \{u, f_{\mathcal{C}_2, f_4}\}) \cup \{z_1, x'\}.$$

Set $h' = (f_2 \setminus \{\bar{u}, l_{\mathcal{C}_2, f_2}\}) \cup \{z_2, x\}$ if $z_2 \in g'_1$ and $h' = (f_2 \setminus \{\bar{u}, f_{\mathcal{C}_2, f_2}, l_{\mathcal{C}_2, f_2}\}) \cup \{y, z_2, x\}$, otherwise. By Claim 4.12, the edges h and h' are red. So $\mathcal{P} h' \mathcal{P}' h$ is a red copy of \mathcal{C}_6^k , a contradiction. \square

Claim 4.14 Let e_i be an arbitrary edge of \mathcal{C}_3 and $n \geq 10$. For every vertices $z \in W$ and $\hat{v} \in (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}) \cup (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\})$ the edge

$$(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, \hat{v}\}$$

is red.

Proof of Claim 4.14. Suppose indirectly that there are vertices $z \in W$ and $\hat{v} \in (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}) \cup (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\})$ so that the edge

$$g = (e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{z, \hat{v}\}$$

is blue. We may assume that $i = 1$ and $\hat{v} \in e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}$. In the rest of the proof, we consider $W \setminus \{z\}$ instead of W when we use Lemmas 3.8, 3.9 and 3.10.

Since $n \geq 10$, we have $|W| \geq 3$. Use Lemma 3.8 for $e_i = e_1$, $f_j = f_1$ and $B = \{w_1, w_2, w_3\} \subseteq W$ to obtain a red path $E_1 = g_1 g'_1$ with the mentioned properties of Lemma 3.8. Let $B' = B \cap E_1$. As mentioned in Lemma 3.8, there are distinct vertices $\bar{v} \in e_1 \setminus E_1$, $\bar{u} \in f_1 \setminus E_1$ and $\bar{v}' \in e_1 \setminus (E_1 \cup \{v_1, v_k, \bar{v}\})$. Set $\tilde{v} = \bar{v}'$, if $\bar{v} = v_1$ and $\tilde{v} = \bar{v}$, otherwise. Clearly $\tilde{v} \in e_1 \setminus (E_1 \cup \{v_1\})$. If $\bar{u} \neq u_1$, then set $g_i = f_i$ for $1 \leq i \leq \frac{n}{2} + 1$. If $\bar{u} = u_1$, then set $g_1 = f_1$ and $g_t = f_{\frac{n}{2}-t+3}$ for $2 \leq t \leq \frac{n}{2} + 1$. Clearly $\mathcal{C}_2 = g_1 g_2 \dots g_{\frac{n}{2}+1}$. With no loss of generality assume that

$$g_i = \{w_1, w_2, \dots, w_k\} + (k-1)(i-1) \pmod{(k-1)(\frac{n}{2}+1)}, \quad i = 1, 2, \dots, \frac{n}{2} + 1.$$

Use Lemma 3.9 for $e_i = e_2$ and $f_j = g_2$ and $\mathcal{E}_1 = E_1$ to obtain two red paths \mathcal{E}_2 and \mathcal{F}_2 of length 4 with the mentioned properties of Lemma 3.9. Now, use Lemma 3.10, $\frac{n}{2} - 4$ times (for $e_i = e_l$ and $f_j = g_l$ where $3 \leq l \leq \frac{n}{2} - 2$), to obtain two red paths $\mathcal{E}_{\frac{n}{2}-2}$ and $\mathcal{F}_{\frac{n}{2}-2}$ of length $n - 4$ with the mentioned properties of Lemma 3.10. Now, use Lemma 3.11 for $v \in \{\bar{v}, \bar{v}'\} \setminus \{\bar{v}\}$ and $u = l_{\mathcal{C}_2, g_{\frac{n}{2}-1}}$ to obtain a red path $\mathcal{E}_{\frac{n}{2}-1}$ of length $n - 2$. With no loss of generality assume that $\mathcal{E}_{\frac{n}{2}-1} = h_1 h_2 \dots h_{n-2}$. Let $x \in (g_1 \setminus \{l_{\mathcal{C}_2, g_1}\}) \cap (h_1 \setminus h_2)$ and $y \in (g_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_2, g_{\frac{n}{2}-1}}\}) \cap (h_{n-2} \setminus h_{n-3})$. By Claim 4.12 the edges

$$g = (g_{\frac{n}{2}} \setminus \{f_{\mathcal{C}_2, g_{\frac{n}{2}}}, l_{\mathcal{C}_2, g_{\frac{n}{2}}}, w_{(k-1)\frac{n}{2}}\}) \{y, z, w_{(k-1)(\frac{n}{2}+1)}\},$$

$$g' = (g_{\frac{n}{2}+1} \setminus \{w_{(k-1)(\frac{n}{2}+1)}, l_{\mathcal{C}_2, g_{\frac{n}{2}+1}}\}) \cup \{x, z\}$$

are red. So $\mathcal{E}_{\frac{n}{2}-1} g g'$ is a red copy of \mathcal{C}_n^k , a contradiction to our assumption. \square

Claim 4.15 *Let e_i and f_j be two arbitrary edges of \mathcal{C}_3 and \mathcal{C}_2 , respectively. Choose $x \in \{f_{\mathcal{C}_2, f_{j-1}}, l_{\mathcal{C}_2, f_j}\}$. For $n = 6$, assume that*

$$(\bar{v}, \hat{v}) \in \left(\{l_{\mathcal{C}_3, e_i}\} \times (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}) \right) \cup \left(\{f_{\mathcal{C}_3, e_i}\} \times (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\}) \right).$$

For $n > 6$, let $\bar{v}, \hat{v} \in e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}$ or $\bar{v}, \hat{v} \in e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\}$. Then, at least one of the edges $(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{x, \bar{v}\}$ or $((e_i \cup \{f_{\mathcal{C}_2, f_{j-1}}, l_{\mathcal{C}_2, f_j}\}) \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}, x\}) \cup \{\hat{v}\}$ is red.

Proof of Claim 4.15. By symmetry we may assume that $e_i = e_1$, $f_j = f_{\frac{n}{2}+1}$ and $x = l_{\mathcal{C}_2, f_{\frac{n}{2}+1}}$. Suppose indirectly that there are vertices \bar{v}, \hat{v} with the mentioned properties so that the edges $(e_1 \setminus \{f_{\mathcal{C}_3, e_1}, l_{\mathcal{C}_3, e_1}\}) \cup \{l_{\mathcal{C}_2, f_{\frac{n}{2}+1}}, \bar{v}\}$ and $(e_1 \setminus \{f_{\mathcal{C}_3, e_1}, l_{\mathcal{C}_3, e_1}\}) \cup \{\hat{v}, f_{\mathcal{C}_2, f_{\frac{n}{2}}}\}$ are blue. With no loss of generality assume that $\bar{v} = l_{\mathcal{C}_3, e_1}$ and $\hat{v} \in e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}$ for $n = 6$ and $\bar{v}, \hat{v} \in e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}$, otherwise. Use Lemma 3.6 for $e = e_1$ (resp. $e = f_1$) to obtain a red path $\mathcal{E}_1 = g_1 g'_1$ (resp. $\mathcal{F}_1 = \bar{g}_1 \bar{g}'_1$) with the mentioned properties of Lemma 3.6 (by putting $i = j = 1$, $v' = v_1$, $v'' = v_k$, $u' = u_1$, $u'' = u_k$, $C = \{v_{k-1}\}$ and $B = \{w_1, w_2\} \subseteq W$). Use Lemma 3.10, $\frac{n}{2} - 3$ times, to obtain two red paths $\mathcal{E}_{\frac{n}{2}-2}$ and $\mathcal{F}_{\frac{n}{2}-2}$ of length $n - 4$ with the properties of Lemma 3.10.

Set $y \in e_{\frac{n}{2}-2} \setminus (\mathcal{E}_{\frac{n}{2}-2} \cup \{f_{\mathcal{C}_3, e_{\frac{n}{2}-2}}, v_{k-1}\})$. Use Lemma 3.7 for $i = \frac{n}{2} - 1$, $j = \frac{n}{2}$, $v' = y$, $v'' = v_{k-1}$, $u' = f_{\mathcal{C}_2, f_{\frac{n}{2}}}$, $u'' = u_{\frac{n}{2}(k-1)+2}$ to obtain a red path $\mathcal{P} = g_{\frac{n}{2}-1} g'_{\frac{n}{2}-1}$ with the properties of Lemma 3.7.

Let $\mathcal{E}_{\frac{n}{2}-2} = h_1 h_2 \dots h_{n-4}$, $\mathcal{P} = h_{n-3} h_{n-2}$, $x' \in (e_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_3, e_{\frac{n}{2}-1}}, l_{\mathcal{C}_3, e_{\frac{n}{2}-1}}, \hat{v}\}) \cap (h_{n-2} \setminus h_{n-3})$, $x'' \in (e_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_3, e_{\frac{n}{2}-1}}, l_{\mathcal{C}_3, e_{\frac{n}{2}-1}}, \hat{v}\}) \cap (h_{n-3} \setminus h_{n-2})$ and $y' \in (f_{\frac{n}{2}-2} \setminus \{f_{\mathcal{C}_2, f_{\frac{n}{2}-2}}\}) \cap (h_{n-4} \setminus h_{n-5})$. By Lemma 3.6, we may assume that $w_1 \in (h_1 \setminus h_2) \cap W$. Since there is no blue copy of \mathcal{C}_n^k , the edges

$$h = (f_{\frac{n}{2}+1} \setminus \{l_{\mathcal{C}_2, f_{\frac{n}{2}+1}}, u_{\frac{n}{2}(k-1)+2}\}) \cup \{x', w_1\}$$

and

$$h' = (f_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_2, f_{\frac{n}{2}-1}}, l_{\mathcal{C}_2, f_{\frac{n}{2}-1}}\}) \cup \{x'', y'\}$$

are red. Therefore, $\mathcal{E}_{\frac{n}{2}-2}h'\mathcal{P}h$ is a red copy of \mathcal{C}_n^k . This contradiction completes the proof of Claim 4.15. \square

Claim 4.16 *Let $n = 6$, $W = \{z_1, z_2\}$ and e_i and f_j be two arbitrary edges of \mathcal{C}_3 and \mathcal{C}_2 , respectively. Let $A = \{a, b\}$ with $f_{\mathcal{C}_2, f_j} \in A$ and $|A \cap W| = 1$. Assume that*

$$(\bar{v}, \hat{v}) \in \left(\{l_{\mathcal{C}_3, e_i}\} \times (e_{i+1} \setminus \{l_{\mathcal{C}_3, e_{i+1}}\}) \right) \cup \left(\{f_{\mathcal{C}_3, e_i}\} \times (e_{i-1} \setminus \{f_{\mathcal{C}_3, e_{i-1}}\}) \right).$$

At least one of the edges $(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{a, \bar{v}\}$ or $(e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{b, \hat{v}\}$ is red.

Proof of Claim 4.16. By symmetry we may assume that $e_i = e_1$, $f_j = f_4$, $a = f_{\mathcal{C}_2, f_4}$, $b = z_1$. Suppose indirectly that there are vertices \bar{v} and \hat{v} with the desired properties so that the edges $(e_1 \setminus \{f_{\mathcal{C}_3, e_1}, l_{\mathcal{C}_3, e_1}\}) \cup \{a, \bar{v}\}$ and $(e_1 \setminus \{f_{\mathcal{C}_3, e_1}, l_{\mathcal{C}_3, e_1}\}) \cup \{\hat{v}, b\}$ are blue. With no loss of generality we may assume that $\bar{v} = l_{\mathcal{C}_3, e_1} = v_k$ and $\hat{v} \in e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}$. Use Lemma 3.6 for $e = e_1$ to obtain a red path $\mathcal{E}_1 = g_1 g'_1$ with the mentioned properties of Lemma 3.6 (by putting $i = j = 1$, $v' = v_1$, $v'' = v_k$, $u' = u_1$, $u'' = u_{k+1}$, $C = \{v_{k-1}\}$ and $B = W = \{z_1, z_2\}$). By symmetry we may assume that either $z_1 \in g_1$ or $z_1 \notin \mathcal{E}_1$. As mentioned in Lemma 3.6, there is a vertex $y \in e_1 \setminus (\mathcal{E}_1 \cup \{v_1, v_{k-1}\})$.

Use Lemma 3.7 for $i = 2$, $j = 3$, $v' = y$, $v'' = v_{k-1}$, $u' = f_{\mathcal{C}_2, f_3}$, $u'' = l_{\mathcal{C}_2, f_3}$ to obtain a red path $\mathcal{P} = g_2 g'_2$ with the properties of Lemma 3.7.

Let $x \in (e_2 \setminus \{v_k, v_1, \hat{v}\}) \cap (g'_2 \setminus g_2)$, $\bar{y} \in (f_1 \setminus \{u_1, u_k\}) \cap (g_1 \setminus g'_1)$, $y' \in (f_1 \setminus \{u_1, u_k\}) \cap (g'_1 \setminus g_1)$ and $y'' \in (f_3 \setminus \{f_{\mathcal{C}_2, f_3}, l_{\mathcal{C}_2, f_3}\}) \cap (g_2 \setminus g'_2)$. Since there is no blue copy of \mathcal{C}_n^k , the edge $h = (f_4 \setminus \{f_{\mathcal{C}_2, f_4}, l_{\mathcal{C}_2, f_4}\}) \cup \{x, y'\}$ is red. If $z_1 \in g_1$, then set

$$h' = (f_2 \setminus \{u_{k+1}, l_{\mathcal{C}_2, f_2}\}) \cup \{z_1, y''\}.$$

Otherwise, set

$$h' = (f_2 \setminus \{f_{\mathcal{C}_2, f_2}, l_{\mathcal{C}_2, f_2}, u_{k+1}\}) \cup \{z_1, \bar{y}, y''\}.$$

By Claim 4.12, the edge h' is red. So, $h\mathcal{E}_1 h'\mathcal{P}$ is a red copy of \mathcal{C}_6^k . This contradiction completes the proof of Claim 4.16. \square

Claim 4.17 *Let $n \geq 10$. There is a red copy of $\mathcal{C}_{\frac{n}{2}-1}^k$, say $\mathcal{C}_4 = h_1 h_2 \dots h_{\frac{n}{2}-1}$, disjoint from \mathcal{C}_1 so that for each $1 \leq i \leq \frac{n}{2} - 1$, $k - 2 \leq |h_i \cap e_i| \leq k - 1$. Moreover, $|h_{\frac{n}{2}-1} \cap e_{\frac{n}{2}-1}| = k - 1$ and*

$$h_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_4, h_{\frac{n}{2}-1}}\} = e_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_3, e_{\frac{n}{2}-1}}\}.$$

Proof of Claim 4.17. Let $n = 4l + 2$ and $W' = V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$. Since $|V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq (k - 1)(\frac{n}{2} + 1) + l + 1$, clearly $|W'| \geq l - 1$. First let $|W'| \geq l$. For $1 \leq i \leq \frac{n}{2} - 1$, set

$$h_i = \begin{cases} (e_i \setminus \{l_{\mathcal{C}_3, e_i}\}) \cup \{z_{\frac{i+1}{2}}\} & \text{if } i \text{ is odd,} \\ (e_i \setminus \{f_{\mathcal{C}_3, e_i}\}) \cup \{z_{\frac{i}{2}}\} & \text{if } i \text{ is even,} \end{cases}$$

where $\{z_1, z_2, \dots, z_l\} \subseteq W'$. By Claim 4.14, $\mathcal{C}_4 = h_1 h_2 \dots h_{\frac{n}{2}-1}$ is a red copy of $\mathcal{C}_{\frac{n}{2}-1}^k$. Now, let $W' = \{z_1, z_2, \dots, z_{l-1}\}$. This is the case only when $|V(\mathcal{C}_1 \cup \mathcal{C}_2)| = (k-1)(\frac{n}{2}+1) + l + 1$. Clearly, $|V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)| = l + 1$. If there is a vertex $x \in f_j \setminus (\{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\} \cup V(\mathcal{C}_1))$, for some edge f_j of \mathcal{C}_2 , then the same argument yields a red $\mathcal{C}_4 = h_1 h_2 \dots h_{\frac{n}{2}-1}$ (consider $W' \cup \{x\}$ instead of W' in the above argument and use Claims 4.10 and 4.14). Therefore, we may assume that each vertex of $V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)$ is a first vertex of f_j for some edge f_j of \mathcal{C}_2 .

If there is an edge f_j of \mathcal{C}_2 so that $\{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\} \subseteq V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)$, then do the following process. Using Claim 4.11, there is a vertex $z \in \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\}$ so that the edge $h_1 = (e_1 \setminus \{v_k\}) \cup \{z\}$ is red. If the edge $(e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}) \cup \{z\}$ is red, then set $h_2 = (e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}) \cup \{z\}$. Otherwise, set $h_2 = (e_2 \setminus \{f_{\mathcal{C}_3, e_2}, l_{\mathcal{C}_3, e_2}\}) \cup \{v_{k-1}, z'\}$, where $z' \in \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\} \setminus \{z\}$. By Claim 4.11, h_2 is red. For $3 \leq i \leq \frac{n}{2} - 1$, set

$$h_i = \begin{cases} (e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{v_{(i-1)(k-1)}, z_{\frac{i-1}{2}}\} & \text{if } i \text{ is odd,} \\ (e_i \setminus \{f_{\mathcal{C}_3, e_i}\}) \cup \{z_{\frac{i}{2}-1}\} & \text{if } i \text{ is even.} \end{cases}$$

By Claim 4.14, h_i 's, $3 \leq i \leq \frac{n}{2} - 1$, are red and clearly $\mathcal{C}_4 = h_1 h_2 \dots h_{\frac{n}{2}-1}$ is a red copy of $\mathcal{C}_{\frac{n}{2}-1}^k$. If each the above cases does not occur, then there is an edge f_j so that $\{f_{\mathcal{C}_2, f_{j-1}}, l_{\mathcal{C}_2, f_j}\} \subseteq V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)$. Similar to the above cases, by Claim 4.15, there is a vertex $z \in \{f_{\mathcal{C}_2, f_{j-1}}, l_{\mathcal{C}_2, f_j}\}$ so that the edge $h_1 = (e_1 \setminus \{v_k\}) \cup \{z\}$ is red. If the edge $(e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}) \cup \{z\}$ is red, then set $h_2 = (e_2 \setminus \{l_{\mathcal{C}_3, e_2}\}) \cup \{z\}$. Otherwise, set $h_2 = (e_2 \setminus \{f_{\mathcal{C}_3, e_2}, l_{\mathcal{C}_3, e_2}\}) \cup \{v_{k-1}, z'\}$, where $z' \in \{f_{\mathcal{C}_2, f_{j-1}}, l_{\mathcal{C}_2, f_j}\} \setminus \{z\}$. By Claim 4.15, h_2 is red. For $3 \leq i \leq \frac{n}{2} - 1$, set

$$h_i = \begin{cases} (e_i \setminus \{f_{\mathcal{C}_3, e_i}, l_{\mathcal{C}_3, e_i}\}) \cup \{v_{(i-1)(k-1)}, z_{\frac{i-1}{2}}\} & \text{if } i \text{ is odd,} \\ (e_i \setminus \{f_{\mathcal{C}_3, e_i}\}) \cup \{z_{\frac{i}{2}-1}\} & \text{if } i \text{ is even.} \end{cases}$$

By Claim 4.14, h_i 's, $3 \leq i \leq \frac{n}{2} - 1$, are red and clearly $\mathcal{C}_4 = h_1 h_2 \dots h_{\frac{n}{2}-1}$ is a red copy of $\mathcal{C}_{\frac{n}{2}-1}^k$. Clearly, in each the above cases, $\mathcal{C}_4 = h_1 h_2 \dots h_{\frac{n}{2}-1}$ is a red copy of $\mathcal{C}_{\frac{n}{2}-1}^k$ disjoint from \mathcal{C}_1 with desired properties. \square

Claim 4.18 *Let $n = 6$. There is a red copy of \mathcal{C}_2^k , say $\mathcal{C}_4 = h_1 h_2$, disjoint from \mathcal{C}_1 so that for $1 \leq i \leq 2$, $k - 2 \leq |h_i \cap e_i| \leq k - 1$. Moreover,*

$$h_1 = (e_1 \setminus \{v_1, v_k\}) \cup \{v, w\}$$

for some $v \in e_2 \setminus \{f_{\mathcal{C}_3, e_2}\}$ and $w \in V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_3)$.

Proof of Claim 4.18. Let $W' = V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$. Since

$$4(k-1) \leq |V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq 4(k-1) + 2,$$

clearly $0 \leq |W'| \leq 2$. First let $|W'| = 2$ and $W' = \{z_1, z_2\}$. If there is a vertex $z \in W'$, say z_1 , so that the edge $(e_1 \setminus \{v_k\}) \cup \{z_1\}$ is blue, then using Claim 4.13, the edge

$$h = (e_1 \setminus \{v_1, v_k\}) \cup \{v_{2k-2}, z_2\}$$

is red. If the edge $(e_2 \setminus \{v_k\}) \cup \{z_2\}$ is red, then set $h_2 = (e_2 \setminus \{v_k\}) \cup \{z_2\}$. Otherwise, set $h_2 = (e_2 \setminus \{v_1, v_k\}) \cup \{v_2, z_1\}$. Using Claim 4.13, the edge h_2 is red. By choosing $h_1 = h$, $\mathcal{C}_4 = h_1 h_2$ is the desired cycle. Therefore, we may assume that for each $z \in W' = \{z_1, z_2\}$, the edge $(e_1 \setminus \{v_k\}) \cup \{z\}$ is red. Now, using Claim 4.13, there is a vertex $z' \in W'$ so that the edge $(e_2 \setminus \{v_k\}) \cup \{z'\}$ is red. With no loss of generality assume that $z' = z_1$. By choosing $h_1 = (e_1 \setminus \{v_k\}) \cup \{z_1\}$ and $h_2 = (e_2 \setminus \{v_k\}) \cup \{z_1\}$, $\mathcal{C}_4 = h_1 h_2$ is the favorable \mathcal{C}_2^k .

Now, assume that $|W'| = 1$ and $W' = \{z\}$. Clearly $|V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)| = 1$. Let $x \in V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)$. If $x \in f_j \setminus \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\}$, for some edge f_j of \mathcal{C}_2 , then using Claim 4.10, the edges $h_1 = (e_1 \setminus \{v_k\}) \cup \{x\}$ and $h_2 = (e_2 \setminus \{v_k\}) \cup \{x\}$ are red and $\mathcal{C}_4 = h_1 h_2$ is the desired cycle. Therefore, we may assume that x is a first vertex of f_j for some edge f_j of \mathcal{C}_2 . If there is a vertex $\bar{x} \in \{x, z\}$ so that the edge $(e_1 \setminus \{v_k\}) \cup \{\bar{x}\}$ is blue, then using Claim 4.16, the edge

$$h' = (e_1 \setminus \{v_1, v_k\}) \cup \{v_{2k-2}, \bar{y}\}$$

is red where $\bar{y} \in \{x, z\} \setminus \{\bar{x}\}$. If the edge $(e_2 \setminus \{v_k\}) \cup \{\bar{y}\}$ is red, then set $h_2 = (e_2 \setminus \{v_k\}) \cup \{\bar{y}\}$. Otherwise, set $h_2 = (e_2 \setminus \{v_1, v_k\}) \cup \{v_2, \bar{x}\}$. Using Claim 4.16, the edge h_2 is red. By choosing $h_1 = h'$, $\mathcal{C}_4 = h_1 h_2$ is the desired cycle. Therefore, we may assume that for each $\bar{x} \in \{x, z\}$, the edge $(e_1 \setminus \{v_k\}) \cup \{\bar{x}\}$ is red. Now, using Claim 4.16, there is a vertex $y \in \{x, z\}$ so that the edge $(e_2 \setminus \{v_k\}) \cup \{y\}$ is red. By choosing $h_1 = (e_1 \setminus \{v_k\}) \cup \{y\}$ and $h_2 = (e_2 \setminus \{v_k\}) \cup \{y\}$, $\mathcal{C}_4 = h_1 h_2$ is the desired \mathcal{C}_2^k .

So we may assume that $|W'| = 0$. This is the case only when $|V(\mathcal{C}_1 \cup \mathcal{C}_2)| = (k-1)(\frac{n}{2}+1)+2$. Clearly, $|V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)| = 2$. Similar to the discussion in the above paragraph, we may assume that each vertex of $V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)$ is a first vertex of f_j for some edge f_j of \mathcal{C}_2 . So we have two following cases.

- (i) There is an edge f_j of \mathcal{C}_2 so that $V(\mathcal{C}_2) \setminus V(\mathcal{C}_1) = \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\}$,
- (ii) There is an edge f_j so that $V(\mathcal{C}_2) \setminus V(\mathcal{C}_1) = \{f_{\mathcal{C}_2, f_{j-1}}, l_{\mathcal{C}_2, f_j}\}$.

Note that, in each cases, we can find the favorable red \mathcal{C}_4 , by an argument that used for case $|W'| = 1$ (use Claim 4.11 for case (i) and Claim 4.15 for case (ii)). So we are done. \square

The proof of the following claim is similar to the proof of Claim 4.9. So we omit it here.

Claim 4.19 *Let h_i and d_j be two arbitrary edges of \mathcal{C}_4 and \mathcal{C}_1 , respectively and $y \in h_i$. For every vertices $x \in d_j \setminus \{f_{\mathcal{C}_1, d_j}\}$ and $x' \in d_{j+2} \setminus \{l_{\mathcal{C}_1, d_{j+2}}\}$, there are distinct vertices $\bar{v}, v' \in h_i \setminus \{f_{\mathcal{C}_4, h_i}, l_{\mathcal{C}_4, h_i}, y\}$, $u \in d_{j-1} \setminus \{f_{\mathcal{C}_1, d_{j-1}}\}$ and $u' \in d_{j+3} \setminus \{l_{\mathcal{C}_1, d_{j+3}}\}$ so that at least one of the edges $g = (V(d_j) \setminus \{f_{\mathcal{C}_1, d_j}, x\}) \cup \{\bar{v}, u\}$ or $g' = (V(d_{j+2}) \setminus \{x', l_{\mathcal{C}_1, d_{j+2}}\}) \cup \{v', u'\}$ is red.*

In the rest of this section, we show that there is a red copy of \mathcal{C}_n^k . First let $n = 6$. By Claim 4.18, there is a red cycle $\mathcal{C}_4 = h_1 h_2$ disjoint from \mathcal{C}_1 where

$h_1 = (e_1 \setminus \{v_1, v_k\}) \cup \{v, w\}$ for some $v \in e_2 \setminus \{f_{\mathcal{C}_3, e_2}\}$ and $w \in V(\mathcal{H}) \setminus (V(\mathcal{C}_1) \cup V(\mathcal{C}_3))$. Since

$$4(k-1) \leq |V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq 4(k-1) + 2,$$

there is a vertex $z \in (d_l \setminus \{f_{\mathcal{C}_1, d_l}, l_{\mathcal{C}_1, d_l}\}) \cap (f_{l'} \setminus \{f_{\mathcal{C}_2, f_{l'}}, l_{\mathcal{C}_2, f_{l'}}\})$ for some $1 \leq l, l' \leq 4$. Using Claim 4.10 the edge

$$g = (h_1 \setminus \{v, w\}) \cup \{z, v_1\} = (e_1 \setminus \{v_k\}) \cup \{z\}$$

is red. With no loss of generality assume that $d_l = d_1$. Now, using Claim 4.19 for $h_i = h_2$, $y = v_1$, $d_j = d_4$, $x = l_{\mathcal{C}_1, d_4}$ and $x' = f_{\mathcal{C}_1, d_2}$, there are distinct vertices $\bar{v}, v' \in h_2 \setminus \{f_{\mathcal{C}_4, h_2}, l_{\mathcal{C}_4, h_2}, y\}$, $u \in d_3 \setminus \{f_{\mathcal{C}_1, d_3}\}$ and $u' \in d_3 \setminus \{l_{\mathcal{C}_1, d_3}\}$ so that at least one of the edges $g' = (d_4 \setminus \{x, f_{\mathcal{C}_1, d_4}\}) \cup \{u, \bar{v}\}$ or $g'' = (d_2 \setminus \{x', l_{\mathcal{C}_1, d_2}\}) \cup \{u', v'\}$ is red. If g' is red, then $g'd_3d_2d_1gh_2$ is a red copy of \mathcal{C}_6^k , a contradiction. If g'' is red, then $gd_1d_4d_3g''h_2$ is a red copy of \mathcal{C}_n^k , a contradiction to our assumption.

Now, let $n > 6$. By Claim 4.17, there is a red cycle $\mathcal{C}_4 = h_1h_2 \dots h_{\frac{n}{2}-1}$, disjoint from \mathcal{C}_1 so that

$$|h_{\frac{n}{2}-1} \cap e_{\frac{n}{2}-1}| = k-1, h_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_4, h_{\frac{n}{2}-1}}\} = e_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_3, e_{\frac{n}{2}-1}}\}$$

and for each $1 \leq i \leq \frac{n}{2} - 1$, $|h_i \cap e_i| \geq k-2$. Since

$$(k-1)(\frac{n}{2} + 1) \leq |V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq (k-1)(\frac{n}{2} + 1) + \lfloor \frac{n}{4} \rfloor + 1,$$

there is a vertex $z \in (d_l \setminus \{f_{\mathcal{C}_1, d_l}, l_{\mathcal{C}_1, d_l}\}) \cap (f_{l'} \setminus \{f_{\mathcal{C}_2, f_{l'}}, l_{\mathcal{C}_2, f_{l'}}\})$ for some $1 \leq l, l' \leq \frac{n}{2} + 1$. Using Claim 4.10 the edge

$$g = (h_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_4, h_{\frac{n}{2}-1}}\}) \cup \{z\} = (e_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_3, e_{\frac{n}{2}-1}}\}) \cup \{z\}$$

is red. Now, using Claim 4.19 for $h_i = h_{\frac{n}{2}-2}$, $d_j = d_{l-1}$, $x = l_{\mathcal{C}_1, d_{l-1}}$ and $x' = f_{\mathcal{C}_1, d_{l+1}}$, there are distinct vertices $\bar{v}, v' \in h_{\frac{n}{2}-2} \setminus \{f_{\mathcal{C}_4, h_{\frac{n}{2}-2}}, l_{\mathcal{C}_4, h_{\frac{n}{2}-2}}\}$, $u \in d_{l-2} \setminus \{f_{\mathcal{C}_1, d_{l-2}}\}$ and $u' \in d_{l+2} \setminus \{l_{\mathcal{C}_1, d_{l+2}}\}$ so that at least one of the edges $g' = (d_{l-1} \setminus \{x, f_{\mathcal{C}_1, d_{l-1}}\}) \cup \{u, \bar{v}\}$ or $g'' = (d_{l+1} \setminus \{x', l_{\mathcal{C}_1, d_{l+1}}\}) \cup \{u', v'\}$ is red. If g' is red, then $g'd_{l-2}d_{l-3} \dots d_1d_{\frac{n}{2}+1}d_{\frac{n}{2}} \dots d_lgh_1h_2 \dots h_{\frac{n}{2}-2}$ is a red copy of \mathcal{C}_n^k , a contradiction. If g'' is red, then

$$gd_ld_{l-1}d_{l-2} \dots d_1d_{\frac{n}{2}+1}d_{\frac{n}{2}} \dots d_{l+2}g''h_{\frac{n}{2}-2}h_{\frac{n}{2}-3} \dots h_2h_1$$

is a red copy of \mathcal{C}_n^k , a contradiction. ■

5 Concluding remarks and open problems

Throughout this paper, we consider $k \geq 8$ for simplicity. We believe that our approach can be used to prove Conjecture 1.5 for $n = m$ and $k = 7$, however much more details are required. Therefore, it would be interesting to investigate Conjecture 1.5 for $n = m$ and $3 < k < 7$.

It is known that Conjecture 1.5 is true for a fixed $m \geq 3$ if it holds for every $m \leq n \leq 2m$ ([15]). So it would be interesting to deduce whether Conjecture 1.5 holds for every $m \leq n \leq 2m$ and $k \geq 4$. It seems that our method can be used to prove Conjecture 1.5 for $m \leq n \leq 2m$ and sufficiently large k , but too much efforts and details are needed.

Another interesting question in this direction is to prove Conjecture 1.5 for small values of k . As we noted in the introduction, the case $k = 3$ is proved in [15] and [14].

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6 Appendix A

Proof the part (i) of Theorem 1.1. Let $\mathcal{H} = \mathcal{K}_{2(k-1)}^k$ is two edge colored red and blue. If \mathcal{H} is monochromatic, then clearly we are done. So we may assume \mathcal{H}_{red} and $\mathcal{H}_{\text{blue}}$ are both non empty. It is easy to see that if a k -uniform complete hypergraph is 2-colored and both colors are used at least once, then there are two edges of distinct colors intersecting in $k-1$ vertices (see Remark 3 of [9]). Thereby, we can select $e = \{v_1, v_2, \dots, v_k\} \in \mathcal{H}_{\text{red}}$ and $f = \{v_2, v_3, \dots, v_{k+1}\} \in \mathcal{H}_{\text{blue}}$. Let $W = V(\mathcal{H}) \setminus \{v_1, v_2, \dots, v_{k+1}\}$. Clearly $|W| = k-3$. Consider the edge $g = \{v_1, v_2, v_{k+1}\} \cup W$. If g is red, then eg is a red copy of \mathcal{C}_2^k . Otherwise, fg is a blue \mathcal{C}_2^k . ■

Proof of Lemma 3.1. Let $\mathcal{C} = e_1 e_2 \dots e_n$ be a copy of \mathcal{C}_n^k in \mathcal{H}_{red} with edges

$$e_i = \{v_1, v_2, \dots, v_k\} + (k-1)(i-1) \pmod{(k-1)n}, \quad i = 1, \dots, n.$$

First assume that n is odd. Let

$$f_i = \begin{cases} (e_{2i-1} \setminus \{l_{\mathcal{C}, e_{2i-1}}\}) \cup \{l_{\mathcal{C}, e_{2i}}\} & 1 \leq i \leq \frac{n+1}{2}, \\ (e_{2i-n-1} \setminus \{l_{\mathcal{C}, e_{2i-n-1}}\}) \cup \{l_{\mathcal{C}, e_{2i-n}}\} & \frac{n+3}{2} \leq i \leq n. \end{cases}$$

Since there is no red copy of \mathcal{C}_{n-1}^k , all f_i 's, $1 \leq i \leq n$ are blue (otherwise, for some i , the edges $f_i e_{2i+1} e_{2i+2} \dots e_n e_1 \dots e_{2i-2}$ form a red \mathcal{C}_{n-1}^k). Clearly $f_1 f_2 \dots f_n$ is a blue copy of \mathcal{C}_n^k .

Now assume that n is even. Let

$$f_i = \begin{cases} (e_{2i-1} \setminus \{v_{(2i-1)(k-1)}, v_{(2i-1)(k-1)+1}\}) \cup \{v_{2i(k-1)}, v_{2i(k-1)+1}\} & 1 \leq i \leq \frac{n}{2} - 1, \\ (e_{n-1} \setminus \{v_{(n-1)(k-1)}, v_{(n-1)(k-1)+1}\}) \cup \{v_{n(k-1)}, v_{n(k-1)+1}\} & i = \frac{n}{2}, \\ \{v_{a(k-1)}, v_{a(k-1)+1}\} \cup (e_{a-1} \setminus \{v_{(a-1)(k-1)}, v_{(a-1)(k-1)+1}\}) & \frac{n}{2} + 1 \leq i \leq n-1, \\ \{v_{k-2}, v_{3(k-1)}, v_{3(k-1)+1}\} \cup (e_2 \setminus \{v_k, v_{2(k-1)}, v_{2(k-1)+1}\}) & i = n, \end{cases}$$

where $1 \leq a \leq n$ and $a = 3 - 2i \pmod{n}$.

It is obvious that all f_i 's are blue. So $f_1 \dots f_n$ is a copy of \mathcal{C}_n^k in $\mathcal{H}_{\text{blue}}$. ■

Proof of Lemma 3.2. Let $\mathcal{C} = e_1 e_2 \dots e_n$ be a copy of \mathcal{C}_n^k in \mathcal{H}_{red} with edges

$$e_i = \{v_1, v_2, \dots, v_k\} + (k-1)(i-1) \pmod{(k-1)n}, \quad i = 1, \dots, n.$$

First assume that $n \equiv 1, 2 \pmod{3}$. Let $f_i = e_{3i-2} \setminus \{l_{\mathcal{C}, e_{3i-2}}\} \cup \{l_{\mathcal{C}, e_{3i}}\}$, $1 \leq i \leq n$ where indices the e_i 's are mod n . Since there is no red copy of \mathcal{C}_{n-2}^k , all f_i 's, $1 \leq i \leq n$ are blue (otherwise, for some i , the edges $f_i e_{3i+1} \dots e_{3i-3}$ form a red \mathcal{C}_{n-2}^k). Clearly $f_1 f_2 \dots f_n$ is a blue copy of \mathcal{C}_n^k .

Now assume that $n \equiv 0 \pmod{3}$. Partition the vertices of e_i into three parts A_i, B_i and C_i with $|A_i| \geq |B_i| \geq |C_i| \geq |A_i| - 1$ so that $f_{C,e_i} \in A_i$ and $l_{C,e_i} \in C_i$. Clearly $|C_i| \geq 2$. Let $v \in C_1 \setminus \{v_k\}$, $v' \in B_1$ and $v'' \in B_2$. Set

$$f_i = \begin{cases} A_{3(i-1)+1} \cup B_{3(i-1)+2} \cup C_{3(i-1)+3} & 1 \leq i \leq \frac{n}{3} - 1, \\ A_{n-2} \cup B_{n-1} \cup (C_n \setminus \{v_1\}) \cup \{v\} & i = \frac{n}{3}, \\ C_{2n+4-3i} \cup B_{2n+3-3i} \cup A_{2n+2-3i} & \frac{n}{3} + 1 \leq i \leq \frac{2n}{3} - 1, \\ C_4 \cup B_3 \cup (A_2 \setminus \{v_k\}) \cup \{v'\} & i = \frac{2n}{3}, \\ C_{3n+5-3i} \cup B_{3n+4-3i} \cup A_{3n+3-3i} & \frac{2n}{3} + 1 \leq i \leq n - 1, \\ C_5 \cup B_4 \cup (A_3 \setminus \{v_{2(k-1)+1}\}) \cup \{v''\} & i = n, \end{cases}$$

where the indices are mod n .

It is obvious that all f_i 's are blue. So $f_1 \dots f_n$ is a copy of \mathcal{C}_n^k in $\mathcal{H}_{\text{blue}}$. \blacksquare

Proof of Lemma 3.6. Suppose for a contradiction that there is no red path \mathcal{P} with the above conditions. By symmetry we may assume that $i = j = 1$, $e_1 = \{v_1, v_2, \dots, v_k\}$, $f_1 = \{u_1, u_2, \dots, u_k\}$ and $C \subseteq \{v_{k-1}\}$. Let $|C| = l \in \{0, 1\}$. Consider an edge $h = E_1 \dot{\cup} W_1 \dot{\cup} F_1$ in $\mathcal{A}_{11} \cup \mathcal{B}_{11}$ so that

- $W_1 \subseteq \{w_1, w_2\}$, $|W_1| = l$.
- $E_1 \subseteq (V(e_1) \setminus (C \cup \{v_1, v_k\})) \cup \{x\}$, $x \in \{v', v''\}$.
- $F_1 \subseteq (V(f_1) \setminus \{u_1, u_k\}) \cup \{y\}$, $y \in \{u', u''\}$.
- $||E_1| - |F_1|| \leq 1$.

Claim 6.1 *The edge h is red.*

Proof of Claim 6.1. By symmetry we may assume that $h \in \mathcal{A}_{11}$, $|F_1| \geq |E_1|$ and $W_1 \subseteq \{w_1\}$. It is easy to see that $|E_1| = \lfloor \frac{k-l}{2} \rfloor$ and $|F_1| = \lceil \frac{k-l}{2} \rceil$. W.l.g. assume that

$$E_1 = \{v', v_2, \dots, v_{\lfloor \frac{k-l}{2} \rfloor}\},$$

$$F_1 = \{u_{k-\lceil \frac{k-l}{2} \rceil+1}, \dots, u_{k-1}, u''\}.$$

Suppose indirectly that the edge $h_1 = h$ is blue. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, using Remark 3.5, every edge in \mathcal{B}_{11} that is disjoint from h_1 is red. Now let $h_2 = (h_1 \setminus \{v_2\}) \cup \{v_{k-1-l}\}$. Clearly $h_2 \in \mathcal{A}_{11}$. If h_2 is red, then set

$$h'_2 = \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_1 \cup C \cup \{v_1, v_k, u_1, u_k, u\}) \right) \cup \{u', v''\}$$

where $u \in f_1 \setminus (h_2 \cup \{u_1, u_k\})$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, $\mathcal{P} = g_1 g_2$ is the desired path where $g_1 = h_2$ and $g_2 = h'_2$ (clearly $w \in e \setminus \mathcal{P}$ where $w = v_2$ for $e = e_1$

and $w = u$ for $e = f_1$). Therefore, we may assume that the edge h_2 is blue. For $2 \leq i \leq \lfloor \frac{k-l}{2} \rfloor$, let $h_i = (h_{i-1} \setminus \{v_i\}) \cup \{v_{k-(i+l)+1}\}$. Assume that j is the maximum $i \in [1, \lfloor \frac{k-l}{2} \rfloor]$ for which h_i is blue. If $j < \lfloor \frac{k-l}{2} \rfloor$, then h_{j+1} is red. Set

$$h'_{j+1} = \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_j \cup C \cup \{v_1, v_k, u_1, u_k, u\}) \right) \cup \{u', v''\},$$

where $u \in f_1 \setminus (h_{j+1} \cup \{u_1, u_k\})$. Clearly, h'_{j+1} is red. Therefore, $\mathcal{P} = g_1 g_2$ is the desired path where $g_1 = h_{j+1}$ and $g_2 = h'_{j+1}$ (clearly $w \in e \setminus \mathcal{P}$ where $w = v_{j+1}$ for $e = e_1$ and $w = u$ for $e = f_1$). So we may assume that $j = \lfloor \frac{k-l}{2} \rfloor$ and hence the edge $h_{\lfloor \frac{k-l}{2} \rfloor} = E' \dot{\cup} W_1 \dot{\cup} F_1$ is blue, where

$$E' = \{v', v_{k-l-(\lfloor \frac{k-l}{2} \rfloor - 1)}, \dots, v_{k-l-2}, v_{k-l-1}\}.$$

Now, set

$$m = \begin{cases} |F_1| - 2 & \text{if } l = 0 \text{ and } k \text{ is odd,} \\ |F_1| - 1 & \text{otherwise.} \end{cases}$$

For $1 \leq i \leq m$, let $h_{\lfloor \frac{k-l}{2} \rfloor + i} = (h_{\lfloor \frac{k-l}{2} \rfloor + i - 1} \setminus \{u_{k-i}\}) \cup \{u_{i+1}\}$. Now, let j be the maximum $i \in [0, m]$ for which $h_{\lfloor \frac{k-l}{2} \rfloor + i}$ is blue. If $j < m$, then $h_{\lfloor \frac{k-l}{2} \rfloor + j + 1}$ is red. Set

$$h'_{\lfloor \frac{k-l}{2} \rfloor + j + 1} = \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + j} \cup C \cup \{v_1, v_k, u_1, u_k, v\}) \right) \cup \{u', v''\},$$

where $v \in e_1 \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + j + 1} \cup \{v_1, v_k\} \cup C)$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, the edge $h'_{\lfloor \frac{k-l}{2} \rfloor + j + 1}$ is red. Therefore, $\mathcal{P} = g_1 g_2$ is the desired path where $g_1 = h_{\lfloor \frac{k-l}{2} \rfloor + j + 1}$ and $g_2 = h'_{\lfloor \frac{k-l}{2} \rfloor + j + 1}$ (clearly $w \in e \setminus \mathcal{P}$ where $w = v$ for $e = e_1$ and $w = u_{k-j-1}$ for $e = f_1$). So we may assume that $j = m$ and hence the edge $h_{\lfloor \frac{k-l}{2} \rfloor + m} = E' \dot{\cup} W_1 \dot{\cup} F'$ is blue, where

$$F' = \begin{cases} \{u_2, \dots, u_{m+1}, u_{\frac{k+1}{2}}, u''\} & \text{if } l = 0 \text{ and } k \text{ is odd,} \\ \{u_2, \dots, u_{m+1}, u''\} & \text{otherwise.} \end{cases}$$

Set $h_{\lfloor \frac{k-l}{2} \rfloor + m + 1} = (h_{\lfloor \frac{k-l}{2} \rfloor + m} \setminus \{v'\}) \cup \{v''\}$. If $h_{\lfloor \frac{k-l}{2} \rfloor + m + 1}$ is red, then set

$$\begin{aligned} h'_{\lfloor \frac{k-l}{2} \rfloor + m + 1} &= \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m} \cup C \cup \{v_1, v_k, u_1, u_k, v\}) \right) \cup \{v'', u'\}, \\ h''_{\lfloor \frac{k-l}{2} \rfloor + m + 1} &= \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m} \cup C \cup \{v_1, v_k, u_1, u_k, u\}) \right) \cup \{v'', u'\}. \end{aligned}$$

where $v \in e_1 \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m + 1} \cup \{v_1, v_k\} \cup C)$ and $u \in f_1 \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m + 1} \cup \{u_1, u_k\})$. For $e = e_1$, set $\mathcal{P} = h_{\lfloor \frac{k-l}{2} \rfloor + m + 1} h'_{\lfloor \frac{k-l}{2} \rfloor + m + 1}$ and for $e = f_1$ set $\mathcal{P} = h_{\lfloor \frac{k-l}{2} \rfloor + m + 1} h''_{\lfloor \frac{k-l}{2} \rfloor + m + 1}$. It is easy to see that \mathcal{P} is the desired path. Hence, we may assume that the edge $h_{\lfloor \frac{k-l}{2} \rfloor + m + 1}$ is blue.

Similarly, we may assume that the edge $h_{\lfloor \frac{k-l}{2} \rfloor + m + 2} = (h_{\lfloor \frac{k-l}{2} \rfloor + m + 1} \setminus \{u''\}) \cup \{u'\}$ is blue. If $l = 0$ and k is even, then clearly $h_{\lfloor \frac{k-l}{2} \rfloor + m + 2}$ is an edge in \mathcal{B}_{11} disjoint

from h_1 . This is impossible, by Remark 3.5. Now we have one of the following cases.

Case 1: $l = 0$ and k is odd.

One can easily check that $u_{\frac{k+1}{2}} \in h_1 \cap h_{\lfloor \frac{k-l}{2} \rfloor + m+2}$ and $v_{\frac{k+1}{2}} \notin h_1 \cup h_{\lfloor \frac{k-l}{2} \rfloor + m+2}$. Let $h_{\lfloor \frac{k-l}{2} \rfloor + m+3} = (h_{\lfloor \frac{k-l}{2} \rfloor + m+2} \setminus \{u_{\frac{k+1}{2}}\}) \cup \{v_{\frac{k+1}{2}}\}$. If the edge $h_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ is red, then set

$$h'_{\lfloor \frac{k-l}{2} \rfloor + m+3} = \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m+2} \cup \{v_1, v_k, u_1, u_k, v\}) \right) \cup \{v', u''\},$$

where $v \in e_1 \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m+3} \cup \{v_1, v_k\})$. It is easy to see that $\mathcal{P} = g_1 g_2$ is the desired path where $g_1 = h'_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ and $g_2 = h'_{\lfloor \frac{k-l}{2} \rfloor + m+3}$. Therefore, we may assume that the edge $h_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ is blue. That is a contradiction to Remark 3.5, since $h_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ is an edge in \mathcal{B}_{11} disjoint from h_1 .

Case 2: $l = 1$.

First let k is even. Clearly, $v_{\frac{k}{2}} \notin h_1 \cup h_{\lfloor \frac{k-l}{2} \rfloor + m+2}$ and $W_1 = \{w_1\}$. Let $h_{\lfloor \frac{k-l}{2} \rfloor + m+3} = (h_{\lfloor \frac{k-l}{2} \rfloor + m+2} \setminus \{w_1\}) \cup \{v_{\frac{k}{2}}\}$. If $h_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ is red, then set

$$\begin{aligned} h'_{\lfloor \frac{k-l}{2} \rfloor + m+3} &= \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m+2} \cup C \cup \{v_1, v_k, u_1, u_k, v\}) \right) \cup \{v', u''\}, \\ h''_{\lfloor \frac{k-l}{2} \rfloor + m+3} &= \left((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m+2} \cup C \cup \{v_1, v_k, u_1, u_k, u\}) \right) \cup \{v', u''\}. \end{aligned}$$

where $v \in e_1 \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m+3} \cup \{v_1, v_k, v_{k-1}\})$ and $u \in f_1 \setminus (h_{\lfloor \frac{k-l}{2} \rfloor + m+3} \cup \{u_1, u_k\})$. Let $g_2 = h_{\lfloor \frac{k-l}{2} \rfloor + m+3}$. It is easy to check that $\mathcal{P} = g_1 g_2$ is the desired path where $g_1 = h'_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ for $e = e_1$ and $g_1 = h''_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ for $e = f_1$. So, we may assume that the edge $h_{\lfloor \frac{k-l}{2} \rfloor + m+3}$ is blue, a contradiction to Remark 3.5.

Now, let k be odd. One can easily see that $u_{\frac{k+1}{2}} \notin h_1 \cup h_{\lfloor \frac{k-l}{2} \rfloor + m+2}$. Similarly, we may assume that the edge $h_{\lfloor \frac{k-l}{2} \rfloor + m+3} = h_{\lfloor \frac{k-l}{2} \rfloor + m+2} \setminus \{w_1\} \cup \{u_{\frac{k+1}{2}}\}$ is blue. That is a contradiction to Remark 3.5. This contradiction completes the proof of Claim 6.1. \square

By Claim 6.1 we can find the favorable red \mathcal{P} , as follow.

First let $l = 1$. Set

$$g_1 = \{v', v_2, \dots, v_{\lfloor \frac{k-1}{2} \rfloor}\} \cup \{w_1\} \cup \{u_{\lfloor \frac{k-1}{2} \rfloor + 2}, \dots, u_{k-1}, u''\}$$

and

$$g_2 = \{v_{\lfloor \frac{k-1}{2} \rfloor + 2}, \dots, v_{k-2}, v''\} \cup \{w_2\} \cup \{u', u_2, u_3, \dots, u_{\lfloor \frac{k-1}{2} \rfloor}, u_{\lfloor \frac{k-1}{2} \rfloor + 2}\}.$$

Clearly, by Claim 6.1, $\mathcal{P} = g_1 g_2$ is the desired path. Now let $l = 0$. Set

$$g_1 = \{v', v_2, \dots, v_{\lfloor \frac{k}{2} \rfloor}\} \cup \{u_{\lfloor \frac{k}{2} \rfloor + 1}, \dots, u_{k-1}, u''\}.$$

For $e = e_1$ let

$$g_2 = \{v_{\lfloor \frac{k}{2} \rfloor}, v_{\lfloor \frac{k}{2} \rfloor + 2}, \dots, v_{k-1}, v''\} \cup \{u', u_2, \dots, u_{\lfloor \frac{k}{2} \rfloor}\}$$

and for $e = f_1$ let

$$g_2 = \{v_{\lfloor \frac{k}{2} \rfloor + 1}, \dots, v_{k-1}, v''\} \cup \{u', u_2, \dots, u_{\lfloor \frac{k}{2} \rfloor - 1}, u_{\lfloor \frac{k}{2} \rfloor + 1}\}.$$

It is easy to see that $\mathcal{P} = g_1 g_2$ is the desired path. So we are done. \blacksquare

Proof of Lemma 3.8. Suppose not. By symmetry we may assume that $i = j = 1$, $e_1 = \{v_1, v_2, \dots, v_k\}$ and $f_1 = \{u_1, u_2, \dots, u_k\}$. Consider an edge $h = E \dot{\cup} W' \dot{\cup} F$ in $\mathcal{A}_{11} \cup \mathcal{B}_{11}$ so that

- $W' \subseteq B$, $|W'| = 1$.
- $E \subseteq V(e_1)$, $F \subseteq V(f_1)$ and $||E| - |F|| \leq 1$.

Claim 6.2 *The edge h is red.*

Proof of Claim 6.2. By symmetry we may assume that $h \in \mathcal{A}_{11}$, $|F| \geq |E|$ and $W' = \{w_1\}$. It is easy to see that $|E| = \lfloor \frac{k-1}{2} \rfloor$ and $|F| = \lceil \frac{k-1}{2} \rceil$. W.l.g. assume that

$$\begin{aligned} E &= \{v_1, v_2, \dots, v_{\lfloor \frac{k-1}{2} \rfloor}\}, \\ F &= \{u_{k - \lceil \frac{k-1}{2} \rceil + 1}, \dots, u_{k-1}, u_k\}. \end{aligned}$$

Suppose indirectly that the edge $h_1 = h$ is blue. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, using Remark 3.5, every edge in \mathcal{B}_{11} that is disjoint from h_1 is red. For $2 \leq i \leq \lfloor \frac{k-1}{2} \rfloor$, let $h_i = (h_{i-1} \setminus \{v_i\}) \cup \{v_{k-i+1}\}$. Now, let j be the maximum $i \in [1, \lfloor \frac{k-1}{2} \rfloor]$ for which h_i is blue. If $j < \lfloor \frac{k-1}{2} \rfloor$, then h_{j+1} is red. Set

$$\begin{aligned} h'_{j+1} &= ((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_j \cup \{u, v\})), \\ h''_{j+1} &= ((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_j \cup \{u, u'\})), \end{aligned}$$

where u, u', v are distinct vertices so that $u, u' \in f_1 \setminus (h_{j+1} \cup \{u_1, u_k\})$ and $v \in e_1 \setminus (h_{j+1} \cup \{v_1, v_k, v_{j+1}\})$. Clearly, h'_{j+1} and h''_{j+1} are red. Set $g_1 = h_{j+1}$, $g'_1 = h'_{j+1}$ and $\bar{g}'_1 = h''_{j+1}$. Then $E_1 = g_1 g'_1$ and $F_1 = g_1 \bar{g}'_1$ are desired paths where $\bar{v} = v_{j+1}$ and $\bar{u} = u$. So we may assume that $j = \lfloor \frac{k-1}{2} \rfloor$ and hence the edge $h_{\lfloor \frac{k-1}{2} \rfloor} = E' \dot{\cup} W' \dot{\cup} F$ is blue, where

$$E' = \{v_1, v_{k - \lfloor \frac{k-1}{2} \rfloor + 1}, \dots, v_{k-2}, v_{k-1}\}.$$

Now, for $1 \leq i \leq \lceil \frac{k-1}{2} \rceil - 1$ let $h_{\lfloor \frac{k-1}{2} \rfloor + i} = (h_{\lfloor \frac{k-1}{2} \rfloor + i - 1} \setminus \{u_{k-i}\}) \cup \{u_{i+1}\}$. In a similar way we can show that the edge $h_{k-2} = E' \dot{\cup} W' \dot{\cup} F'$ is blue, where

$$F' = \{u_2, u_3, \dots, u_{\lceil \frac{k-1}{2} \rceil}, u_k\}.$$

Now, let $h_{k-1} = (h_{k-2} \setminus \{v_1, w_1\}) \cup \{v_k, w_3\}$. If h_{k-1} is red, then set

$$\begin{aligned} h'_{k-1} &= ((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{k-2} \cup \{u, v\})), \\ h''_{k-1} &= ((e_1 \cup f_1 \cup \{w_2\}) \setminus (h_{k-2} \cup \{u, u'\})). \end{aligned}$$

where u, u', v are distinct vertices so that $u, u' \in f_1 \setminus (h_{k-1} \cup \{u_1, u_k\})$ and $v \in e_1 \setminus (h_{k-1} \cup \{v_1, v_k\})$. Clearly, h'_{k-1} and h''_{k-1} are red. Set $g_1 = h_{k-1}$, $g'_1 = h'_{k-1}$ and

$\overline{g}'_1 = h''_{k-1}$. Then $E_1 = g_1 g'_1$ and $F_1 = g_1 \overline{g}'_1$ are desired paths where $\overline{v} = v_1$ and $\overline{u} = u$. Hence, we may assume that the edge h_{k-1} is blue. Similarly, we may assume that the edge $h_k = (h_{k-1} \setminus \{u_k\}) \cup \{u_1\}$ is blue. This is a contradiction to Remark 3.5. This contradiction completes the proof of Claim 6.2. \square

Now, we can find favorable paths as follows:

Let

$$\begin{aligned} g_1 &= \{v_1, v_2, \dots, v_{\lfloor \frac{k-1}{2} \rfloor}\} \cup \{w_1\} \cup \{u_{k-\lceil \frac{k-1}{2} \rceil+1}, \dots, u_{k-1}, u_k\}, \\ g'_1 &= \{v_{\lfloor \frac{k-1}{2} \rfloor}, v_{\lfloor \frac{k-1}{2} \rfloor+3}, \dots, v_k\} \cup \{w_2\} \cup \{u_1, \dots, u_{\lfloor \frac{k-1}{2} \rfloor}\}, \\ \overline{g}'_1 &= \{v_{\lfloor \frac{k-1}{2} \rfloor+2}, \dots, v_k\} \cup \{w_2\} \cup \{u_1, \dots, u_{\lfloor \frac{k-1}{2} \rfloor-1}, u_{k-\lceil \frac{k-1}{2} \rceil+1}\}. \end{aligned}$$

Using Claim 6.2, the edges g_1 , g'_1 and \overline{g}'_1 are red and so $E_1 = g_1 g'_1$ and $F_1 = g_1 \overline{g}'_1$ are desired paths where $\overline{v} = v_{\lfloor \frac{k-1}{2} \rfloor+1}$, $\overline{u} = u_{\lfloor \frac{k-1}{2} \rfloor+1}$ and $B' = \{w_1, w_2\}$. \blacksquare

Proof of Lemma 3.9. By symmetry we may assume that $i = j = 2$. Suppose for a contradiction that there is no red paths E_2 and F_2 with desired properties. Let $x \in (e_1 \setminus \{v_1\}) \cap (g'_1 \setminus g_1)$. Assume that $h = E \cup F$ is an edge in \mathcal{A}_{22} so that

$$\begin{aligned} E &= \{x, v_{k+1}, \dots, v_{(k-1)+\lfloor \frac{k}{2} \rfloor}\}, \\ F &= \{u_{2k-\lceil \frac{k}{2} \rceil}, \dots, u_{2k-2}, u_{2k-1}\}. \end{aligned}$$

Claim 6.3 *The edge h is red.*

Proof of Claim 6.3. Suppose indirectly that the edge $h_1 = h$ is blue. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, using Remark 3.5, every edge in \mathcal{B}_{22} that is disjoint from h_1 is red. Now let $h_2 = (h_1 \setminus \{x\}) \cup \{v_{2k-1}\}$. If h_2 is red, then set

$$\begin{aligned} h'_2 &= ((e_2 \cup f_2 \cup \{w\}) \setminus (h_1 \cup \{f_{\mathcal{C}_1, e_2}, f_{\mathcal{C}_2, f_2}, \tilde{v}\})) \cup \{\overline{u}\}, \\ h''_2 &= ((e_2 \cup f_2 \cup \{w\}) \setminus (h_1 \cup \{f_{\mathcal{C}_1, e_2}, f_{\mathcal{C}_2, f_2}, \tilde{u}\})) \cup \{\overline{u}\}, \end{aligned}$$

where $\tilde{v} \in e_2 \setminus (h_2 \cup \{f_{\mathcal{C}_1, e_2}, l_{\mathcal{C}_1, e_2}\})$ and $\tilde{u} \in f_2 \setminus (h_2 \cup \{f_{\mathcal{C}_2, f_2}, l_{\mathcal{C}_2, f_2}\})$. Set $g_2 = h'_2$, $\overline{g}_2 = h''_2$ and $g'_2 = \overline{g}'_2 = h_2$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, $E_2 = g_2 g'_2$ and $F_2 = \overline{g}_2 \overline{g}'_2$ are desired paths. Therefore, we may assume that the edge h_2 is blue. For $3 \leq l \leq \lfloor \frac{k}{2} \rfloor + 1$, let $h_l = (h_{l-1} \setminus \{v_{k-2+l}\}) \cup \{v_{2k-l+1}\}$. Assume that l' is the maximum $l \in [2, \lfloor \frac{k}{2} \rfloor + 1]$ for which h_l is blue. If $l' < \lfloor \frac{k}{2} \rfloor + 1$, then $h_{l'+1}$ is red. Set

$$h'_{l'+1} = ((e_2 \cup f_2 \cup \{w\}) \setminus (h_{l'} \cup \{f_{\mathcal{C}_1, e_2}, f_{\mathcal{C}_2, f_2}, \tilde{u}\})) \cup \{\overline{u}, \overline{v}\},$$

where $\tilde{u} \in f_2 \setminus (h_{l'+1} \cup \{f_{\mathcal{C}_2, f_2}, l_{\mathcal{C}_2, f_2}\})$. Clearly, $h'_{l'+1}$ is red. Set $g_2 = \overline{g}_2 = h'_{l'+1}$ and $g'_2 = \overline{g}'_2 = h_{l'+1}$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, $E_2 = g_2 g'_2$ and $F_2 = \overline{g}_2 \overline{g}'_2$ are desired paths. Also, $\mathcal{E}_2 = \mathcal{E}_1 E_2$ and $\mathcal{F}_2 = \mathcal{E}_1 F_2$ are two red paths of length 4. Therefore, we may assume that $l' = \lfloor \frac{k}{2} \rfloor + 1$ and hence the edge $h_{\lfloor \frac{k}{2} \rfloor+1} = E' \cup F$ is blue, where

$$E' = \{v_{2k-\lfloor \frac{k}{2} \rfloor}, \dots, v_{2k-2}, v_{2k-1}\}.$$

Now, set $m = \lfloor \frac{k}{2} \rfloor - 1$. For $1 \leq l \leq m$, let $h_{\lfloor \frac{k}{2} \rfloor+l+1} = (h_{\lfloor \frac{k}{2} \rfloor+l} \setminus \{u_{2(k-1)-l+1}\}) \cup \{u_{k+l}\}$. Let l' be the maximum $l \in [0, m]$ for which $h_{\lfloor \frac{k}{2} \rfloor+l+1}$ is blue. Similar to the

above argument we can show that $l' = m$ and hence the edge $h_{\lfloor \frac{k}{2} \rfloor + m + 1} = E' \cup F'$ is blue, where

$$F' = \begin{cases} \{u_{k+1}, \dots, u_{k+m}, u_{2k-1}\} & \text{if } k \text{ is even,} \\ \{u_{k+1}, \dots, u_{k+m}, u_{k+m+1}, u_{2k-1}\} & \text{if } k \text{ is odd.} \end{cases}$$

Let $h_{\lfloor \frac{k}{2} \rfloor + m + 2} = (h_{\lfloor \frac{k}{2} \rfloor + m + 1} \setminus \{u_{2k-1}\}) \cup \{\bar{u}\}$. If $h_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is red, then set

$$h'_{\lfloor \frac{k}{2} \rfloor + m + 2} = ((e_2 \cup f_2 \cup \{w\}) \setminus (h_{\lfloor \frac{k}{2} \rfloor + m + 1} \cup \{f_{\mathcal{C}_1, e_2}, f_{\mathcal{C}_2, f_2}, l_{\mathcal{C}_2, f_2}, \tilde{v}\})) \cup \{\bar{u}, \bar{v}\},$$

where $\tilde{v} \in e_2 \setminus (h_{\lfloor \frac{k}{2} \rfloor + m + 2} \cup \{f_{\mathcal{C}_1, e_2}, l_{\mathcal{C}_1, e_2}\})$. Since there is no blue copy of $\mathcal{C}_{l_1 + l_2}$, the edge $h'_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is red. Set $g_2 = \bar{g}_2 = h'_{\lfloor \frac{k}{2} \rfloor + m + 2}$ and $g'_2 = \bar{g}'_2 = h_{\lfloor \frac{k}{2} \rfloor + m + 2}$. Therefore, $E_2 = g_2 g'_2$ and $F_2 = \bar{g}_2 \bar{g}'_2$ are desired paths. So we may assume that the edge $h_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is blue.

If k is even, then clearly $h_{\lfloor \frac{k}{2} \rfloor + m + 2}$ is an edge in \mathcal{B}_{22} disjoint from h_1 . This is impossible, by Remark 3.5. Now we may assume that k is odd. One can easily see that $v_{k+\frac{k-1}{2}} \notin h_1 \cup h_{\lfloor \frac{k}{2} \rfloor + m + 2}$ and $u_{k+\frac{k-1}{2}} \in h_1 \cap h_{\lfloor \frac{k}{2} \rfloor + m + 2}$. Similarly, we can show that the edge $h_{\lfloor \frac{k}{2} \rfloor + m + 3} = h_{\lfloor \frac{k}{2} \rfloor + m + 2} \setminus \{u_{k+\frac{k-1}{2}}\} \cup \{v_{k+\frac{k-1}{2}}\}$ is blue. That is a contradiction to Remark 3.5. This contradiction completes the proof of Claim 6.3. \square

Now, let $h' = \bar{E} \cup \bar{F}$ be an edge in \mathcal{B}_{22} so that

$$\begin{aligned} \bar{E} &= \bar{E}' \cup \{l_{\mathcal{C}_1, e_2}\}, \quad |\bar{E}| = \lceil \frac{k}{2} \rceil, \quad \bar{E}' \subseteq e_2 \setminus \{f_{\mathcal{C}_1, e_2}, l_{\mathcal{C}_1, e_2}\}, \\ \bar{F} &= \bar{F}' \cup \{\bar{u}\}, \quad |\bar{F}| = \lfloor \frac{k}{2} \rfloor, \quad \bar{F}' \subseteq f_2 \setminus \{f_{\mathcal{C}_2, f_2}, l_{\mathcal{C}_2, f_2}\}. \end{aligned}$$

By an argument similar to the proof of Claim 6.3 we can show the following.

Claim 6.4 *The edge h' is red.*

Now, by choosing edges h and h' appropriately as follows, we can find red paths E_2 and F_2 with desired properties. Let

$$g_2 = \bar{g}_2 = h = \{x, v_{k+1}, \dots, v_{(k-1)+\lfloor \frac{k}{2} \rfloor}\} \cup \{u_{2k-\lceil \frac{k}{2} \rceil}, \dots, u_{2k-2}, u_{2k-1}\}.$$

Set $E_2 = g_2 g'_2$ and $F_2 = \bar{g}_2 \bar{g}'_2$ where

$$\begin{aligned} g'_2 &= \{v_{k-1+\lfloor \frac{k}{2} \rfloor}, v_{k+\lfloor \frac{k}{2} \rfloor+1}, \dots, v_{2k-2}, l_{\mathcal{C}_1, e_2}\} \cup \{\bar{u}, u_{k+1}, \dots, u_{k+\lfloor \frac{k}{2} \rfloor-1}\}, \\ \bar{g}'_2 &= \{v_{k+\lfloor \frac{k}{2} \rfloor}, \dots, v_{2k-2}, l_{\mathcal{C}_1, e_2}\} \cup \{\bar{u}, u_{k+1}, \dots, u_{k+\lfloor \frac{k}{2} \rfloor-2}, u_{k+\lfloor \frac{k}{2} \rfloor}\}. \end{aligned}$$

Using Claims 6.3 and 6.4, $E_2 = g_2 g'_2$ and $F_2 = \bar{g}_2 \bar{g}'_2$ are desired paths. Note that, $\mathcal{E}_2 = \mathcal{E}_1 E_2$ and $\mathcal{F}_2 = \mathcal{E}_1 F_2$ are two red paths of length 4. This is a contradiction to our assumption and so we are done. \blacksquare

Proof of Lemma 3.10. Suppose for a contradiction that there is no red paths E_i and F_i with desired properties. Let $u' \in f_{i-1} \setminus (F_{i-1} \cup \{fc_{2,f_{i-1}}\})$, $v \in (e_{i-1} \setminus \{fc_{1,e_{i-1}}\}) \cap (\overline{g}_{i-1} \setminus \overline{g}_{i-1})$ and $w \in W \setminus (\bigcup_{j=1}^{i-1} B_j)$. Assume that $h = E \cup F$ is an edge in \mathcal{A}_{ii} so that

$$\begin{aligned} E &= \overline{E} \cup \{v\}, \quad |E| = \lfloor \frac{k}{2} \rfloor, \quad \overline{E} \subseteq e_i \setminus \{fc_{1,e_i}, lc_{1,e_i}\}, \\ F &= \overline{F} \cup \{lc_{2,f_i}\}, \quad |F| = \lceil \frac{k}{2} \rceil, \quad \overline{F} \subseteq f_i \setminus \{fc_{2,f_i}, lc_{2,f_i}\}. \end{aligned}$$

Claim 6.5 *The edge h is red.*

Proof of Claim 6.5. By changing the indices we may assume that

$$\begin{aligned} E &= \{v, v_{(k-1)(i-1)+2}, \dots, v_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor}\}, \\ F &= \{u_{(k-1)i-\lceil \frac{k}{2} \rceil+2}, \dots, u_{(k-1)i}, lc_{2,f_i}\}. \end{aligned}$$

Suppose indirectly that the edge $h_1 = h$ is blue. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, using Remark 3.5, every edge in \mathcal{B}_{ii} that is disjoint from h_1 is red. Now let $h_2 = (h_1 \setminus \{v_{(k-1)(i-1)+2}\}) \cup \{v_{(k-1)i}\}$. If h_2 is red, then set

$$h'_2 = \left((e_i \cup f_i \cup \{w\}) \setminus (h_1 \cup \{fc_{1,e_i}, fc_{2,f_i}, lc_{2,f_i}, \bar{u}\}) \right) \cup \{u'\},$$

where $\bar{u} \in f_i \setminus (h_2 \cup \{fc_{2,f_i}, lc_{2,f_i}\})$. Set $g_i = \overline{g}_i = h_2$ and $g'_i = \overline{g}'_i = h'_2$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, $E_i = g_i g'_i$ and $F_i = \overline{g}_i \overline{g}'_i$ are desired paths (clearly, $v_{(k-1)(i-1)+2} \in e_i \setminus (E_i \cup \{fc_{1,e_i}\})$, $\bar{u} \in f_i \setminus (F_i \cup \{fc_{2,f_i}\})$). So for $\mathcal{P} = \mathcal{F}_{i-1}$, $\mathcal{P}E_i$ and $\mathcal{P}F_i$ are two red paths of length $2i$. Therefore, we may assume that the edge h_2 is blue. For $2 \leq l \leq \lfloor \frac{k}{2} \rfloor$, let $h_l = (h_{l-1} \setminus \{v_{(k-1)(i-1)+l}\}) \cup \{v_{(k-1)i-l+2}\}$. Assume that l' is the maximum $l \in [1, \lfloor \frac{k}{2} \rfloor]$ for which h_l is blue. If $l' < \lfloor \frac{k}{2} \rfloor$, then $h_{l'+1}$ is red. Set

$$h'_{l'+1} = \left((e_i \cup f_i \cup \{w\}) \setminus (h_{l'} \cup \{fc_{1,e_i}, fc_{2,f_i}, lc_{2,f_i}, \bar{u}\}) \right) \cup \{u'\},$$

where $\bar{u} \in f_i \setminus (h_{l'+1} \cup \{fc_{2,f_i}, lc_{2,f_i}\})$. Clearly, $h'_{l'+1}$ is red. Set $g_i = \overline{g}_i = h_{l'+1}$ and $g'_i = \overline{g}'_i = h'_{l'+1}$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, $E_i = g_i g'_i$ and $F_i = \overline{g}_i \overline{g}'_i$ are desired paths (clearly $v_{(k-1)(i-1)+l'+1} \in e_i \setminus (E_i \cup \{fc_{1,e_i}\})$ and $\bar{u} \in f_i \setminus (F_i \cup \{fc_{2,f_i}\})$). So for $\mathcal{P} = \mathcal{F}_{i-1}$, $\mathcal{P}E_i$ and $\mathcal{P}F_i$ are two red paths of length $2i$. Therefore, we may assume that $l' = \lfloor \frac{k}{2} \rfloor$ and hence the edge $h_{\lfloor \frac{k}{2} \rfloor} = E'' \cup F$ is blue, where

$$E'' = \{v, v_{(k-1)i-\lfloor \frac{k}{2} \rfloor+2}, \dots, v_{(k-1)i-1}, v_{(k-1)i}\}.$$

Now, set $m = \lfloor \frac{k}{2} \rfloor - 1$. For $1 \leq l \leq m$, let $h_{\lfloor \frac{k}{2} \rfloor+l} = (h_{\lfloor \frac{k}{2} \rfloor+l-1} \setminus \{u_{(k-1)i-l+1}\}) \cup \{u_{(k-1)(i-1)+l+1}\}$. Now, let l' be the maximum $l \in [0, m]$ for which $h_{\lfloor \frac{k}{2} \rfloor+l}$ is blue. If $l' < m$, then $h_{\lfloor \frac{k}{2} \rfloor+l'+1}$ is red. Set

$$h'_{\lfloor \frac{k}{2} \rfloor+l'+1} = \left((e_i \cup f_i \cup \{w\}) \setminus (h_{\lfloor \frac{k}{2} \rfloor+l'} \cup \{fc_{1,e_i}, fc_{2,f_i}, lc_{2,f_i}, \bar{v}\}) \right) \cup \{u'\},$$

where $\bar{v} \in e_i \setminus (h_{\lfloor \frac{k}{2} \rfloor+l'+1} \cup \{fc_{1,e_i}, lc_{1,e_i}\})$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, the edge $h'_{\lfloor \frac{k}{2} \rfloor+l'+1}$ is red. Set $g_i = \overline{g}_i = h_{\lfloor \frac{k}{2} \rfloor+l'+1}$ and $g'_i = \overline{g}'_i = h'_{\lfloor \frac{k}{2} \rfloor+l'+1}$. Therefore,

$E_i = g_i g'_i$ and $F_i = \bar{g}_i \bar{g}'_i$ are desired paths (clearly $\bar{v} \in e_i \setminus (E_i \cup \{f_{\mathcal{C}_1, e_i}\})$ and $u_{(k-1)i-l'} \in f_i \setminus (F_i \cup \{f_{\mathcal{C}_2, f_i}\})$) and for $\mathcal{P} = \mathcal{F}_{i-1}$, $\mathcal{P}E_i$ and $\mathcal{P}F_i$ are two red paths of length $2i$. So we may assume that $l' = m$ and hence the edge $h_{\lfloor \frac{k}{2} \rfloor + m} = E'' \cup F''$ is blue, where

$$F'' = \begin{cases} \{u_{(k-1)(i-1)+2}, \dots, u_{(k-1)(i-1)+m+1}, l_{\mathcal{C}_2, f_i}\} & \text{if } k \text{ is even,} \\ \{u_{(k-1)(i-1)+2}, \dots, u_{(k-1)(i-1)+m+1}, u_{(k-1)(i-1)+m+2}, l_{\mathcal{C}_2, f_i}\} & \text{if } k \text{ is odd.} \end{cases}$$

Let $h_{\lfloor \frac{k}{2} \rfloor + m+1} = (h_{\lfloor \frac{k}{2} \rfloor + m} \setminus \{v\}) \cup \{l_{\mathcal{C}_1, e_i}\}$. If $h_{\lfloor \frac{k}{2} \rfloor + m+1}$ is red, then set

$$\begin{aligned} h'_{\lfloor \frac{k}{2} \rfloor + m+1} &= ((e_i \cup f_i \cup \{w\}) \setminus (h_{\lfloor \frac{k}{2} \rfloor + m} \cup \{f_{\mathcal{C}_1, e_i}, f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}, \bar{v}\})) \cup \{u\}, \\ h''_{\lfloor \frac{k}{2} \rfloor + m+1} &= ((e_i \cup f_i \cup \{w\}) \setminus (h_{\lfloor \frac{k}{2} \rfloor + m} \cup \{f_{\mathcal{C}_1, e_i}, f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}, \bar{u}\})) \cup \{u\}. \end{aligned}$$

where $\bar{v} \in e_i \setminus (h_{\lfloor \frac{k}{2} \rfloor + m+1} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})$, $\bar{u} \in f_j \setminus (h_{\lfloor \frac{k}{2} \rfloor + m+1} \cup \{f_{\mathcal{C}_2, f_j}, l_{\mathcal{C}_2, f_j}\})$ and $u \in (f_{j-1} \setminus \{f_{\mathcal{C}_2, f_{j-1}}\}) \cap (\bar{g}'_{i-1} \setminus \bar{g}_{i-1})$. Set $g'_i = \bar{g}'_i = h_{\lfloor \frac{k}{2} \rfloor + m+1}$, $g_i = h'_{\lfloor \frac{k}{2} \rfloor + m+1}$ and $\bar{g}_i = h''_{\lfloor \frac{k}{2} \rfloor + m+1}$. It is easy to see that $E_i = g_i g'_i$ and $F_i = \bar{g}_i \bar{g}'_i$ are the desired paths and hence for $\mathcal{P} = \mathcal{F}_{i-1}$, $\mathcal{E}_i = \mathcal{P}E_i$ and $\mathcal{F}_i = \mathcal{P}F_i$ are two favorable red paths of length $2i$. So, we may assume that the edge $h_{\lfloor \frac{k}{2} \rfloor + m+1}$ is blue.

Now, let $h_{\lfloor \frac{k}{2} \rfloor + m+2} = (h_{\lfloor \frac{k}{2} \rfloor + m+1} \setminus \{l_{\mathcal{C}_2, f_i}\}) \cup \{u'\}$. If $h_{\lfloor \frac{k}{2} \rfloor + m+2}$ is red, then set

$$h'_{\lfloor \frac{k}{2} \rfloor + m+2} = ((e_i \cup f_i \cup \{w\}) \setminus (h_{\lfloor \frac{k}{2} \rfloor + m+1} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}, f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}, \bar{v}\})) \cup \{v, u'\},$$

where $\bar{v} \in e_i \setminus (h_{\lfloor \frac{k}{2} \rfloor + m+2} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})$. Since there is no blue copy of $\mathcal{C}_{l_1+l_2}$, the edge $h'_{\lfloor \frac{k}{2} \rfloor + m+2}$ is red. Set $g_i = \bar{g}_i = h'_{\lfloor \frac{k}{2} \rfloor + m+2}$ and $g'_i = \bar{g}'_i = h_{\lfloor \frac{k}{2} \rfloor + m+2}$. Therefore, $E_i = g_i g'_i$ and $F_i = \bar{g}_i \bar{g}'_i$ are the desired path (clearly $\bar{v} \in e_i \setminus (E_i \cup \{f_{\mathcal{C}_1, e_i}\})$ and $l_{\mathcal{C}_2, f_i} \in f_i \setminus (F_i \cup \{f_{\mathcal{C}_2, f_i}\})$). So for $\mathcal{P} = \mathcal{F}_{i-1}$, $\mathcal{E}_i = \mathcal{P}E_i$ and $\mathcal{F}_i = \mathcal{P}F_i$ are two red paths of length $2i$. So we may assume that the edge $h_{\lfloor \frac{k}{2} \rfloor + m+2}$ is blue.

If k is even, then clearly $h_{\lfloor \frac{k}{2} \rfloor + m+2}$ is an edge in \mathcal{B}_{ii} disjoint from h_1 . This is impossible, by Remark 3.5. Now we may assume that k is odd. One can easily see that $v_{(k-1)(i-1)+\frac{k+1}{2}} \notin h_1 \cup h_{\lfloor \frac{k}{2} \rfloor + m+2}$ and $u_{(k-1)(i-1)+\frac{k+1}{2}} \in h_1 \cap h_{\lfloor \frac{k}{2} \rfloor + m+2}$. Similarly, we can show that the edge $h_{\lfloor \frac{k}{2} \rfloor + m+3} = h_{\lfloor \frac{k}{2} \rfloor + m+2} \setminus \{u_{(k-1)(i-1)+\frac{k+1}{2}}\} \cup \{v_{(k-1)(i-1)+\frac{k+1}{2}}\}$ is blue. That is a contradiction to Remark 3.5. This contradiction completes the proof of Claim 6.5. \square

Now, let $h' = E' \cup F'$ be an edge in \mathcal{B}_{ii} so that

$$\begin{aligned} E' &= \bar{E}' \cup \{l_{\mathcal{C}_1, e_i}\}, \quad |E'| = \lceil \frac{k}{2} \rceil, \quad \bar{E}' \subseteq e_i \setminus \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\}, \\ F' &= \bar{F}' \cup \{u'\}, \quad |F'| = \lfloor \frac{k}{2} \rfloor, \quad \bar{F}' \subseteq f_i \setminus \{f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}\}. \end{aligned}$$

Claim 6.6 *The edge h' is red.*

Proof of Claim 6.6. By changing the indices we may assume that

$$\begin{aligned} E' &= \{v_{(k-1)i-\lceil \frac{k}{2} \rceil + 2}, \dots, v_{(k-1)i}, l_{\mathcal{C}_1, e_i}\}, \\ F' &= \{u', u_{(k-1)(i-1)+2}, \dots, u_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor}\}. \end{aligned}$$

Suppose indirectly that the edge $h_1 = h'$ is blue. Let $m = \lfloor \frac{k}{2} \rfloor$. For $2 \leq l \leq m$, let $h_l = (h_{l-1} \setminus \{v_{(k-1)i-l+2}\}) \cup \{v_{(k-1)(i-1)+l}\}$. Similar to the proof of Claim 6.5, we can show that the edge $h_m = E'' \cup F'$ is blue where

$$E'' = \begin{cases} \{v_{(k-1)(i-1)+2}, \dots, v_{(k-1)(i-1)+m}, l_{\mathcal{C}_1, e_i}\} & \text{if } k \text{ is even,} \\ \{v_{(k-1)(i-1)+2}, \dots, v_{(k-1)(i-1)+m}, v_{(k-1)(i-1)+m+1}, l_{\mathcal{C}_1, e_i}\} & \text{if } k \text{ is odd.} \end{cases}$$

Now, set $m' = \lfloor \frac{k}{2} \rfloor - 1$. For $1 \leq l \leq m'$, let $h_{m+l} = (h_{m+l-1} \setminus \{u_{(k-1)(i-1)+l+1}\}) \cup \{u_{(k-1)i+1-l}\}$. By an argument similar to the proof of Claim 6.5 we may assume that the edge $h_{m+m'} = E'' \cup F''$ is blue, where

$$F'' = \{u', u_{(k-1)i-m'+1}, \dots, u_{(k-1)i}\}.$$

Let $h_{m+m'+1} = (h_{m+m'} \setminus \{u'\}) \cup \{l_{\mathcal{C}_2, f_i}\}$. If $h_{m+m'+1}$ is red, then set

$$\begin{aligned} h'_{m+m'+1} &= \left((e_i \cup f_i \cup \{w\}) \setminus (h_{m+m'} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}, f_{\mathcal{C}_2, f_i}, \bar{v}\}) \right) \cup \{v\}, \\ h''_{m+m'+1} &= \left((e_i \cup f_j \cup \{w\}) \setminus (h_{m+m'} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}, f_{\mathcal{C}_2, f_i}, \bar{u}\}) \right) \cup \{v\}. \end{aligned}$$

where $\bar{v} \in e_i \setminus (h_{m+m'+1} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})$ and $\bar{u} \in f_i \setminus (h_{m+m'+1} \cup \{f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}\})$. Set $g'_i = \bar{g}'_i = h_{m+m'+1}$, $g_i = h'_{m+m'+1}$ and $\bar{g}_i = h''_{m+m'+1}$. It is easy to see that $E_i = g_i g'_i$ and $F_i = \bar{g}_i \bar{g}'_i$ are desired paths. Clearly for $\mathcal{P} = \mathcal{F}_{i-1}$, $\mathcal{E}_i = \mathcal{P}E_i$ and $\mathcal{F}_i = \mathcal{P}F_i$ are two red paths of length $2i$. Hence, we may assume that the edge $h_{m+m'+1}$ is blue.

Let $v' \in e_{i-1} \setminus (E_{i-1} \cup \{f_{\mathcal{C}_1, e_{i-1}}\})$ and $h_{m+m'+2} = (h_{m+m'+1} \setminus \{l_{\mathcal{C}_1, e_i}\}) \cup \{v'\}$. If $h_{m+m'+2}$ is red, then set

$$\begin{aligned} h'_{m+m'+2} &= \left((e_i \cup f_i \cup \{w\}) \setminus (h_{m+m'+1} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}, f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}, \bar{v}\}) \right) \cup \{\hat{u}, v'\}, \\ h''_{m+m'+2} &= \left((e_i \cup f_i \cup \{w\}) \setminus (h_{m+m'+1} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}, f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}, \bar{u}\}) \right) \cup \{\hat{u}, v'\}. \end{aligned}$$

where $\bar{v} \in e_i \setminus (h_{m+m'+2} \cup \{f_{\mathcal{C}_1, e_i}, l_{\mathcal{C}_1, e_i}\})$, $\bar{u} \in f_i \setminus (h_{m+m'+2} \cup \{f_{\mathcal{C}_2, f_i}, l_{\mathcal{C}_2, f_i}\})$ and $\hat{u} \in (f_{i-1} \setminus \{f_{\mathcal{C}_2, f_{i-1}}\}) \cap (g'_{i-1} \setminus g_{i-1})$. Set $g'_i = \bar{g}'_i = h_{m+m'+2}$, $g_i = h'_{m+m'+2}$ and $\bar{g}_i = h''_{m+m'+2}$. Then $E_i = g_i g'_i$ and $F_i = \bar{g}_i \bar{g}'_i$ are desired paths (note that $\bar{v} \in e_i \setminus (E_i \cup \{f_{\mathcal{C}_1, e_i}\})$ and $\bar{u} \in f_i \setminus (F_i \cup \{f_{\mathcal{C}_2, f_i}\})$) and so for $\mathcal{P} = \mathcal{E}_{i-1}$, $\mathcal{E}_i = \mathcal{P}E_i$ and $\mathcal{F}_i = \mathcal{P}F_i$ are two red paths of length $2i$, a contradiction to our assumption. Hence, we may assume that the edge $h_{m+m'+2}$ is blue.

If k is even, then clearly $h_{m+m'+2}$ is an edge in \mathcal{A}_{ii} disjoint from h_1 . This is impossible, by Remark 3.5. Now we may assume that k is odd. One can easily see that $v_{(k-1)(i-1)+\frac{k+1}{2}} \in h_1 \cap h_{m+m'+2}$ and $u_{(k-1)(i-1)+\frac{k+1}{2}} \notin h_1 \cup h_{m+m'+2}$. Similarly, we can show that the edge

$$h_{m+m'+3} = (h_{m+m'+2} \setminus \{v_{(k-1)(i-1)+\frac{k+1}{2}}\}) \cup \{u_{(k-1)(i-1)+\frac{k+1}{2}}\}$$

is blue. Hence, $h_{m+m'+3}$ is an edge in \mathcal{A}_{ii} disjoint from h_1 , that is a contradiction to Remark 3.5. This contradiction completes the proof of Claim 6.6. \square

Now, by choosing edges h and h' appropriately as follows, we can find red paths E_i and F_i with desired properties. Let

$$g_i = \{v, v_{(k-1)(i-1)+2}, \dots, v_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor}\} \cup \{u_{(k-1)i-\lceil \frac{k}{2} \rceil+2}, \dots, u_{(k-1)i}, l_{\mathcal{C}_2, f_i}\}.$$

Set $E_i = g_i g'_i$ and $F_i = \overline{g_i} \overline{g'_i}$ where $\overline{g_i} = g_i$ and

$$\begin{aligned} g'_i &= \{v_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor}, v_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor+2}, v_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor+3}, \dots, v_{(k-1)i}, l_{\mathcal{C}_1, e_i}\} \\ &\quad \cup \{u', u_{(k-1)(i-1)+2}, \dots, u_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor}\}, \\ \overline{g'_i} &= \{v_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor+1}, \dots, v_{(k-1)i}, l_{\mathcal{C}_1, e_i}\} \\ &\quad \cup \{u', u_{(k-1)(i-1)+2}, \dots, u_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor-1}, u_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor+1}\}. \end{aligned}$$

Using Claims 6.5 and 6.6, E_i and F_i are desired paths (clearly, $v_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor+1} \in e_i \setminus (E_i \cup \{f_{\mathcal{C}_1, e_i}\})$ and $u_{(k-1)(i-1)+\lfloor \frac{k}{2} \rfloor} \in f_i \setminus (F_i \cup \{f_{\mathcal{C}_2, f_i}\})$). Note that, $\mathcal{E}_i = \mathcal{P}E_i$ and $\mathcal{F}_i = \mathcal{P}F_i$ are two red paths of length $2i$ where $\mathcal{P} = \mathcal{F}_{i-1}$. This is a contradiction to our assumption and so we are done. \blacksquare

Proof of Lemma 3.12. Let $\mathcal{C}_1 = e_1 e_2 \dots e_{\frac{n-1}{2}}$ and $\mathcal{C}_2 = f_1 f_2 \dots f_{\frac{n+1}{2}}$ be copies of $\mathcal{C}_{\frac{n-1}{2}}^k$ and $\mathcal{C}_{\frac{n+1}{2}}^k$ in $\mathcal{H}_{\text{blue}}$ with edges

$$e_i = \{v_1, v_2, \dots, v_k\} + (k-1)(i-1) \pmod{(k-1)(\frac{n-1}{2})}, \quad i = 1, 2, \dots, \frac{n-1}{2}$$

and

$$f_i = \{u_1, u_2, \dots, u_k\} + (k-1)(i-1) \pmod{(k-1)(\frac{n+1}{2})}, \quad i = 1, 2, \dots, \frac{n+1}{2}.$$

Let $W = V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)$. First assume that $n = 5$. Since there is no blue copy of \mathcal{C}_5^k , consider $i = j = 1$, $e = f_1$, $B = W = \{w_1, w_2\}$, $C = \{v_{k-1}\}$, $v' = v_1$, $u' = u_1$, $v'' = v_k$, $u'' = u_k$ and use Lemma 3.6, to obtain a red path $\mathcal{P} = g_1 g_2$ with the mentioned properties in Lemma 3.6. Clearly, $V(\mathcal{P}) \subseteq (V(e_1 \cup f_1) \setminus \{v_{k-1}\}) \cup W$ and $g_1 \cap W \neq \emptyset$. With no loss of generality assume that $w_1 \in W \cap g_1$. Set

$$h = (e_2 \setminus \{v_k, v_{2k-2}, v_1\}) \cup \{x_1, u_{3k-3}, y_1\},$$

and

$$h' = (f_3 \setminus \{u_{3k-4}, u_{3k-3}, u_1\}) \cup \{w_1, v_{2k-2}, x_2\},$$

where $x_1 \in (g_1 \setminus g_2) \cap (e_1 \setminus \{v_1\})$, $y_1 \in (g_2 \setminus g_1) \cap (f_1 \setminus \{u_k\})$ and $x_2 \in (g_2 \setminus g_1) \cap (e_1 \setminus \{v_k\})$. Since there is no blue copy of \mathcal{C}_5^k , at least one of $\mathcal{P}h$ or $\mathcal{P}h'$ is a red \mathcal{C}_3^k and so we are done.

Now, let $n \geq 7$. We find a red $\mathcal{C}_{\frac{n+1}{2}}^k$ as follows.

Step 1: Put $i = j = 1$, $e = f_1$, $C = \emptyset$, $v' = v_1$, $u' = u_1$, $v'' = v_k$, $u'' = u_k$ and use Lemma 3.6, to obtain a red path $\mathcal{P} = g_1 g_2$ with the mentioned properties in Lemma 3.6. So $V(\mathcal{P}_1) \subseteq V(e_1 \cup f_1) \cup W$ and $|V(\mathcal{P}_1) \cap W| \leq 1$.

Step 2: For $n = 7$ go to step 3. Otherwise, do the following process $\frac{n-7}{2}$ times. For $1 \leq i \leq \frac{n-7}{2}$, let

$$h_{i+1} = (e_{i+1} \setminus \{f_{\mathcal{C}_1, e_{i+1}}, v_{(i+1)(k-1)}, l_{\mathcal{C}_1, e_{i+1}}\}) \cup \{x_{i+1}, u_{(i+1)(k-1)}, l_{\mathcal{C}_2, f_{i+1}}\},$$

and

$$h'_{i+1} = (f_{i+1} \setminus \{f_{\mathcal{C}_2, f_{i+1}}, u_{(i+1)(k-1)}, l_{\mathcal{C}_2, f_{i+1}}\}) \cup \{y_{i+1}, v_{(i+1)(k-1)}, l_{\mathcal{C}_1, e_{i+1}}\},$$

where $x_{i+1} \in (g_{i+1} \setminus g_i) \cap (e_i \setminus \{f_{\mathcal{C}_1, e_i}\})$ and $y_{i+1} \in (g_{i+1} \setminus g_i) \cap (f_i \setminus \{f_{\mathcal{C}_2, f_i}\})$ are vertices with maximum indices. One can easily check that at least one of h_{i+1} or h'_{i+1} , say g_{i+2} , is red. Set $\mathcal{P}_{i+1} = \mathcal{P}_i g_{i+2}$. Clearly \mathcal{P}_{i+1} is a red path of length $i + 2$.

Step 3: Let

$$h_{\frac{n-3}{2}} = (e_{\frac{n-1}{2}} \setminus \{f_{\mathcal{C}_1, e_{\frac{n-1}{2}}}, v_{\frac{n-3}{2}(k-1)+2}, v_1\}) \cup \{x_{\frac{n-3}{2}}, u_{\frac{n-3}{2}(k-1)}, l_{\mathcal{C}_2, f_{\frac{n-3}{2}}}\},$$

and

$$h'_{\frac{n-3}{2}} = (f_{\frac{n-3}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-3}{2}}}, u_{\frac{n-3}{2}(k-1)}, l_{\mathcal{C}_2, f_{\frac{n-3}{2}}}\}) \cup \{y_{\frac{n-3}{2}}, f_{\mathcal{C}_1, e_{\frac{n-1}{2}}}, v_{\frac{n-3}{2}(k-1)+2}\},$$

where $x_{\frac{n-3}{2}} \in (g_1 \setminus g_2) \cap (e_1 \setminus \{v_k\})$ is the vertex with minimum indices and $y_{\frac{n-3}{2}} \in (g_{\frac{n-3}{2}} \setminus g_{\frac{n-5}{2}}) \cap (f_{\frac{n-5}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-5}{2}}}\})$ is the vertex with maximum indices. If $h_{\frac{n-3}{2}}$ is red, then set

$$h_{\frac{n-1}{2}} = (e_{\frac{n-3}{2}} \setminus \{f_{\mathcal{C}_1, e_{\frac{n-3}{2}}}, v_{\frac{n-3}{2}(k-1)}, l_{\mathcal{C}_1, e_{\frac{n-3}{2}}}\}) \cup \{x_{\frac{n-1}{2}}, u_{\frac{n-5}{2}(k-1)+k-2}, l_{\mathcal{C}_2, f_{\frac{n-3}{2}}}\},$$

and

$$h'_{\frac{n-1}{2}} = (f_{\frac{n-3}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-3}{2}}}, u_{\frac{n-5}{2}(k-1)+k-2}, l_{\mathcal{C}_2, f_{\frac{n-3}{2}}}\}) \cup \{y_{\frac{n-1}{2}}, v_{\frac{n-3}{2}(k-1)}, l_{\mathcal{C}_1, e_{\frac{n-3}{2}}}\},$$

where $x_{\frac{n-1}{2}} \in (g_{\frac{n-3}{2}} \setminus g_{\frac{n-5}{2}}) \cap (e_{\frac{n-5}{2}} \setminus \{f_{\mathcal{C}_1, e_{\frac{n-5}{2}}}\})$ and $y_{\frac{n-1}{2}} \in (g_{\frac{n-3}{2}} \setminus g_{\frac{n-5}{2}}) \cap (f_{\frac{n-5}{2}} \setminus \{f_{\mathcal{C}_2, f_{\frac{n-5}{2}}}\})$. So clearly at least one of $\mathcal{P}_{\frac{n-5}{2}} h_{\frac{n-1}{2}} h_{\frac{n-3}{2}}$ or $\mathcal{P}_{\frac{n-5}{2}} h'_{\frac{n-1}{2}} h_{\frac{n-3}{2}}$ is a red $\mathcal{C}_{\frac{n+1}{2}}^k$. Now, assume that $h_{\frac{n-3}{2}}$ is blue. If $h'_{\frac{n-3}{2}}$ is blue, then

$$e_1 e_2 \dots e_{\frac{n-3}{2}} h'_{\frac{n-3}{2}} f_{\frac{n-5}{2}} f_{\frac{n-7}{2}} \dots f_1 f_{\frac{n+1}{2}} f_{\frac{n-1}{2}} h_{\frac{n-3}{2}},$$

is a blue copy of \mathcal{C}_n^k , a contradiction. So $h'_{\frac{n-3}{2}}$ is red. Therefore, set

$$h_{\frac{n-1}{2}} = (e_{\frac{n-1}{2}} \setminus \{v_{\frac{n-3}{2}(k-1)+2}, v_1\}) \cup \{u_{\frac{n+1}{2}(k-1)}, y_{\frac{n-1}{2}}\},$$

and

$$h'_{\frac{n-1}{2}} = (f_{\frac{n+1}{2}} \setminus \{u_{\frac{n+1}{2}(k-1)}, u_1\}) \cup \{v_{\frac{n-3}{2}(k-1)+2}, x_{\frac{n-1}{2}}\},$$

where $x_{\frac{n-1}{2}} \in (g_1 \setminus g_2) \cap (e_1 \setminus \{v_k\})$ and $y_{\frac{n-1}{2}} \in (g_1 \setminus g_2) \cap (f_1 \setminus \{u_k\})$ are vertices with minimum indices. Since at least one of $h_{\frac{n-1}{2}}$ and $h'_{\frac{n-1}{2}}$, say $\bar{h}_{\frac{n-1}{2}}$, is red, then $\mathcal{P}_{\frac{n-5}{2}} h'_{\frac{n-3}{2}} \bar{h}_{\frac{n-1}{2}}$ is a red $\mathcal{C}_{\frac{n+1}{2}}^k$, which completes the proof. ■

Proof of Lemma 3.13. Let $\mathcal{C}_1 = e_1 e_2 \dots e_{\frac{n}{2}-1}$ and $\mathcal{C}_2 = f_1 f_2 \dots f_{\frac{n}{2}+1}$ be copies of $\mathcal{C}_{\frac{n}{2}-1}^k$ and $\mathcal{C}_{\frac{n}{2}+1}^k$ in $\mathcal{H}_{\text{blue}}$ with edges

$$\begin{aligned} e_i &= \{v_1, v_2, \dots, v_k\} + (k-1)(i-1) \pmod{(k-1)(\frac{n}{2}-1)}, \quad i = 1, 2, \dots, \frac{n}{2}-1, \\ f_i &= \{u_1, u_2, \dots, u_k\} + (k-1)(i-1) \pmod{(k-1)(\frac{n}{2}+1)}, \quad i = 1, 2, \dots, \frac{n}{2}+1, \end{aligned}$$

and $W = V(\mathcal{H}) \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)$. First let $n \geq 8$. We can find a red copy of $\mathcal{C}_{\frac{n}{2}+1}^k$ as follows.

Step 1: Put $i = j = 1$, $B = \{w_1, w_2\} \subseteq W$, $C = \{v_{k-1}\}$, $v' = v_1$, $u' = u_1$, $v'' = v_k$ and $u'' = u_k$. Then use Lemma 3.6 for $e = e_1$ (resp. $e = f_1$), to obtain a red path E_1 (resp. F_1) of length 2 with the mentioned properties in Lemma 3.6. Now, use Lemma 3.10 for $i = 2$, to obtain two red paths E_2 and F_2 of length 2 with the mentioned properties in Lemma 3.10. So $\mathcal{E}_2 = \mathcal{P}E_2$ is a red path of length 4 for some $\mathcal{P} \in \{E_1, F_1\}$. Assume that $\mathcal{P}_2 = \mathcal{E}_2 = g_1 g_2 g_3 g_4$. With no loss of generality we may assume that $w_1 \in g_1 \setminus g_2$.

Step 2: If $n = 8$ go to step 3. Otherwise, do the following process $\frac{n}{2} - 4$ times. For $1 \leq i \leq \frac{n}{2} - 4$, let

$$\begin{aligned} h_{i+2} &= (e_{i+2} \setminus \{f_{\mathcal{C}_1, e_{i+2}}, v_{(i+2)(k-1)}, l_{\mathcal{C}_1, e_{i+2}}\}) \cup \\ &\quad \{x_{i+2}, u_{(i+2)(k-1)}, l_{\mathcal{C}_2, f_{i+2}}\} \end{aligned}$$

and

$$\begin{aligned} h'_{i+2} &= (f_{i+2} \setminus \{f_{\mathcal{C}_2, f_{i+2}}, u_{(i+2)(k-1)}, l_{\mathcal{C}_2, f_{i+2}}\}) \cup \\ &\quad \{y_{i+2}, v_{(i+2)(k-1)}, l_{\mathcal{C}_1, e_{i+2}}\}, \end{aligned}$$

where $x_{i+2} \in (g_{i+3} \setminus g_{i+2}) \cap (e_{i+1} \setminus \{f_{\mathcal{C}_1, e_{i+1}}\})$ and $y_{i+1} \in (g_{i+3} \setminus g_{i+2}) \cap (f_{i+1} \setminus \{f_{\mathcal{C}_2, f_{i+1}}\})$ are vertices with maximum indices. Since at least one of h_{i+2} and h'_{i+2} , say g_{i+4} , is red, clearly $\mathcal{P}_{i+2} = \mathcal{P}_{i+1} g_{i+4}$ is a red path of length $i + 4$.

Step 3: Set

$$\begin{aligned} h_{\frac{n}{2}-1} &= (e_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_1, e_{\frac{n}{2}-1}}, v_{(\frac{n}{2}-1)(k-1)}, v_1\}) \cup \\ &\quad \{x_{\frac{n}{2}-1}, w_1, l_{\mathcal{C}_2, f_{\frac{n}{2}-1}}\} \end{aligned}$$

and

$$\begin{aligned} h'_{\frac{n}{2}-1} &= (f_{\frac{n}{2}-1} \setminus \{f_{\mathcal{C}_2, f_{\frac{n}{2}-1}}, u_{(\frac{n}{2}-1)(k-1)}, l_{\mathcal{C}_2, f_{\frac{n}{2}-1}}\}) \cup \\ &\quad \{y_{\frac{n}{2}-1}, v_{(\frac{n}{2}-1)(k-1)}, x'_{\frac{n}{2}-1}\}, \end{aligned}$$

so that $x'_{\frac{n}{2}-1} \in (g_1 \setminus g_2) \cap (e_1 \setminus \{v_k\})$ is a vertex with minimum index and $x_{\frac{n}{2}-1} \in (g_{\frac{n}{2}} \setminus g_{\frac{n}{2}-1}) \cap (e_{\frac{n}{2}-2} \setminus \{fc_1, e_{\frac{n}{2}-2}\})$, $y_{\frac{n}{2}-1} \in (g_{\frac{n}{2}} \setminus g_{\frac{n}{2}-1}) \cap (f_{\frac{n}{2}-2} \setminus \{fc_2, f_{\frac{n}{2}-2}\})$ are vertices with maximum indices. Clearly, at least one of $\mathcal{P}_{\frac{n}{2}-2}h_{\frac{n}{2}-1}$ or $\mathcal{P}_{\frac{n}{2}-2}h'_{\frac{n}{2}-1}$ is a red $\mathcal{C}_{\frac{n}{2}+1}^k$.

Now, let $n = 6$. Suppose indirectly that there is no blue copy of \mathcal{C}_4^k . Put $i = j = 1$, $e = f_1$, $B = W = \{w_1, w_2\}$, $C = \{v_{k-1}\}$, $v' = v_1$, $u' = u_1$, $v'' = v_k$, $u'' = u_k$ and use Lemma 3.6, to obtain a red path $\mathcal{P} = g_1g_2$ with so that $V(\mathcal{P}) \subseteq (V(e_1 \cup f_1) \setminus \{v_{k-1}\}) \cup B$, $(g_1 \setminus g_2) \cap B \neq \emptyset$ and there is a vertex $y \in f_1 \setminus (V(\mathcal{P}) \cup \{u_1\})$. With no loss of generality assume that $w_1 \in g_1 \setminus g_2$. Let $x \in (e_1 \setminus \{v_1\}) \cap (g_2 \setminus g_1)$, $x' \in (e_1 \setminus \{v_k\}) \cap (g_1 \setminus g_2)$ and $h = E \cup F$ is an edge in \mathcal{A}_{22} so that

$$\begin{aligned} E &= \{x, v_{k+1}, \dots, v_{k+\lfloor \frac{k}{2} \rfloor - 1}\}, \\ F &= \{u_{2k-\lceil \frac{k}{2} \rceil}, \dots, u_{2k-1}\}. \end{aligned}$$

Claim 6.7 *The edge h is red.*

Proof of Claim 6.7. Suppose indirectly that the edge $h_1 = h$ is blue. Since there is no blue copy of \mathcal{C}_6^k , using Remark 3.5, every edge in \mathcal{B}_{22} that is disjoint from h_1 is red. For $2 \leq l \leq \lfloor \frac{k}{2} \rfloor$, let $h_l = (h_{l-1} \setminus \{v_{k-1+l}\}) \cup \{v_{2(k-1)-l+2}\}$. Assume that l' is the maximum $l \in [1, \lfloor \frac{k}{2} \rfloor]$ for which h_l is blue. If $l' < \lfloor \frac{k}{2} \rfloor$, then $h_{l'+1}$ is red. Set

$$h'_{l'+1} = ((e_2 \cup f_2) \setminus (h_{l'} \cup \{v_k, v_1, u_k, l_{\mathcal{C}_2, f_2}\})) \cup \{y, x'\}.$$

Since there is no blue copy of \mathcal{C}_6^k , the edge $h'_{l'+1}$ is red and $\mathcal{P}h_{l'+1}h'_{l'+1}$ is a red copy of \mathcal{C}_4^k . So we may assume that $l' = \lfloor \frac{k}{2} \rfloor$ and the edge $h_{\lfloor \frac{k}{2} \rfloor} = E' \cup F$ is blue, where

$$E' = \{x, v_{2k-\lfloor \frac{k}{2} \rfloor}, \dots, v_{2k-3}, v_{2k-2}\}.$$

Now, set $m = \lfloor \frac{k}{2} \rfloor - 1$. For $1 \leq l \leq m$, let $h_{\lfloor \frac{k}{2} \rfloor + l} = (h_{\lfloor \frac{k}{2} \rfloor + l - 1} \setminus \{u_{2(k-1)-l+1}\}) \cup \{u_{k+l}\}$. Now, let l' be the maximum $l \in [0, m]$ for which $h_{\lfloor \frac{k}{2} \rfloor + l}$ is blue. If $l' < m$, then $h_{\lfloor \frac{k}{2} \rfloor + l' + 1}$ is red. Set

$$h'_{\lfloor \frac{k}{2} \rfloor + l' + 1} = ((e_2 \cup f_2) \setminus (h_{\lfloor \frac{k}{2} \rfloor + l'} \cup \{v_1, v_k, u_k, l_{\mathcal{C}_2, f_2}\})) \cup \{y, x'\}.$$

Since there is no blue copy of \mathcal{C}_6^k , the edge $h'_{\lfloor \frac{k}{2} \rfloor + l' + 1}$ is red and $\mathcal{P}h_{\lfloor \frac{k}{2} \rfloor + l' + 1}h'_{\lfloor \frac{k}{2} \rfloor + l' + 1}$ is a red copy of \mathcal{C}_4^k . So we may assume that $l' = m$ and hence the edge $h_{\lfloor \frac{k}{2} \rfloor + m} = E' \cup F'$ is blue, where

$$F' = \begin{cases} \{u_{k+1}, \dots, u_{k+m}, u_{2k-1}\} & k \text{ is even,} \\ \{u_{k+1}, \dots, u_{k+m}, u_{k+m+1}, u_{2k-1}\} & k \text{ is odd.} \end{cases}$$

Let $y' \in (g_2 \setminus g_1) \cap (f_1 \setminus \{u_1\})$. Now, consider the edge $h_{\lfloor \frac{k}{2} \rfloor + m + 1} = (h_{\lfloor \frac{k}{2} \rfloor + m} \setminus \{x, u_{2k-1}\}) \cup \{v_{k-1}, y'\}$. If $h_{\lfloor \frac{k}{2} \rfloor + m + 1}$ is red, then set

$$h'_{\lfloor \frac{k}{2} \rfloor + m + 1} = ((e_2 \cup f_2) \setminus (h_{\lfloor \frac{k}{2} \rfloor + m} \cup \{v_k, v_1, u_k, l_{\mathcal{C}_2, f_2}, \hat{u}\})) \cup \{y, v_{k-1}, w_1\},$$

where $\hat{u} \in f_2 \setminus (h_{\lfloor \frac{k}{2} \rfloor + m} \cup \{u_k, u_{2k-1}\})$. Clearly, $\mathcal{P}h_{\lfloor \frac{k}{2} \rfloor + m+1}h'_{\lfloor \frac{k}{2} \rfloor + m+1}$ is a red copy of \mathcal{C}_4^k , a contradiction to our assumption. So we may assume that $h_{\lfloor \frac{k}{2} \rfloor + m+1}$ is blue. If k is even, then clearly $h_{\lfloor \frac{k}{2} \rfloor + m+1}$ is an edge in \mathcal{B}_{22} disjoint from h_1 , that is a contradiction to Remark 3.5.

Now, we may assume that k is odd. One can easily check that $v_{k+\lfloor \frac{k}{2} \rfloor} \notin h_1 \cup h_{\lfloor \frac{k}{2} \rfloor + m+1}$ and $u_{k+\lfloor \frac{k}{2} \rfloor} \in h_1 \cap h_{\lfloor \frac{k}{2} \rfloor + m+1}$. By an argument similar to the above, we can show that the edge $h_{\lfloor \frac{k}{2} \rfloor + m+2} = (h_{\lfloor \frac{k}{2} \rfloor + m+1} \setminus \{u_{k+\lfloor \frac{k}{2} \rfloor}\}) \cup \{v_{k+\lfloor \frac{k}{2} \rfloor}\}$ is blue. That is a contradiction to Remark 3.5, since $h_{\lfloor \frac{k}{2} \rfloor + m+2}$ is a blue edge in \mathcal{B}_{22} disjoint from h_1 . This contradiction completes the proof of Claim 6.7. \square

Now, let $h' = \overline{E} \cup \overline{F}$ be an edge in \mathcal{B}_{22} so that

$$\begin{aligned}\overline{E} &= \{v_{2k-\lceil \frac{k}{2} \rceil}, \dots, v_{2k-2}, x'\}, \\ \overline{F} &= \{y, u_{k+1}, \dots, u_{k+\lfloor \frac{k}{2} \rfloor-1}\}.\end{aligned}$$

Claim 6.8 *The edge h' is red.*

Suppose indirectly that the edge $h_1 = h'$ is blue. Let $m = m' = \lfloor \frac{k}{2} \rfloor - 1$. Similar to the proof of Claim 6.7, we can show that the edge $h_{m+m'+1} = \overline{E'} \cup \overline{F'}$ is blue where

$$\overline{F'} = \{y, u_{2k-\lfloor \frac{k}{2} \rfloor}, \dots, u_{2k-2}\}$$

and

$$\overline{E'} = \begin{cases} \{v_{k+1}, \dots, v_{k+m}, x'\} & k \text{ is even,} \\ \{v_{k+1}, \dots, v_{k+m}, v_{k+m+1}, x'\} & k \text{ is odd.} \end{cases}$$

Now, consider the edge $h_{m+m'+2} = (h_{m+m'+1} \setminus \{y\}) \cup \{u_{2k-1}\}$. If $h_{m+m'+2}$ is red, then set

$$h'_{m+m'+2} = ((e_2 \cup f_2) \setminus (h_{m+m'+1} \cup \{v_k, v_1, u_k\})) \cup \{x\}.$$

It is easy to see that $h'_{m+m'+2}$ is red and $\mathcal{P}h'_{m+m'+2}h_{m+m'+2}$ is a red copy of \mathcal{C}_4^k , a contradiction to our assumptions. So we may assume that $h_{m+m'+2}$ is blue.

Now, let

$$h_{m+m'+3} = (h_{m+m'+2} \setminus \{x', v_{k+\lceil \frac{k}{2} \rceil-1}\}) \cup \{v_{k-1}, w_1\}.$$

If $h_{m+m'+3}$ is red, then set

$$h'_{m+m'+3} = ((e_2 \cup f_2) \setminus (h_{m+m'+2} \cup \{v_k, v_1, u_k\})) \cup \{v_{k-1}, y'\},$$

where $y' \in (f_1 \setminus \{u_1\}) \cap (g_2 \setminus g_1)$. Since there is no blue copy of \mathcal{C}_k^6 , then $h'_{m+m'+3}$ is red and hence $\mathcal{P}h'_{m+m'+3}h_{m+m'+3}$ is a red copy of \mathcal{C}_4^k , a contradiction to our assumption. So we may assume that the edge $h_{m+m'+3}$ is blue. Notice that $h_{m+m'+3}$ is a blue edge in \mathcal{A}_{22} disjoint from h_1 , a contradiction to Remark 3.5. This contradiction

completes the proof of Claim 6.8. \square

Now, let

$$\begin{aligned} h &= \{x, v_{k+1}, \dots, v_{k+\lfloor \frac{k}{2} \rfloor - 1}\} \cup \{u_{2k-\lceil \frac{k}{2} \rceil}, \dots, u_{2k-1}\}, \\ h' &= \{v_{k+\lfloor \frac{k}{2} \rfloor - 1}, v_{k+\lfloor \frac{k}{2} \rfloor + 1}, \dots, v_{2k-2}, x'\} \cup \{u', u_{k+1}, \dots, u_{k+\lfloor \frac{k}{2} \rfloor - 1}\}. \end{aligned}$$

Using Claims 6.7 and 6.8, the edges h and h' are red. So $\mathcal{P}hh'$ is a red copy of \mathcal{C}_4^k . This contradicts our assumption. So we are done. \blacksquare